Published by Institute of Physics Publishing for SISSA

Received: *October 21, 2007*
Accepted: *November 15, 2007*
Published: *November 30, 2007*

Wrapped magnetized branes: two alternative descriptions?*

P. Di Vecchia
The Niels Bohr Institute, Blegdamsvej 17,
DK-2100 Copenhagen Ø, Denmark, and
NORDITA, Roslagstullsbacken 23,
SE-10691 Stockholm, Sweden
E-mail: divecchi@nbi.dk

A. Liccardo, R. Marotta and F. Pezzella
Dipartimento di Scienze Fisiche, Università di Napoli and
INFN, Sezione di Napoli Complesso Universitario Monte S. Angelo,
ed. 6, via Cintia, I-80126 Napoli, Italy
E-amil: liccardo@na.infn.it, lmarotta@na.infn.it, pezzella@na.infn.it

I. Pesando
Dipartimento di Fisica Teorica, Università di Torino and
INFN, Sezione di Torino, via P. Giuria 1, I-10125, Torino, Italy
E-mail: ipesando@to.infn.it

Abstract: We discuss two inequivalent ways for describing magnetized D-branes wrapped N times on a torus T^2. The first one is based on a non-abelian gauge bundle $\mathrm{U}(N)$, while the second one is obtained by means of a Narain T-duality transformation acting on a theory with non-magnetized branes. We construct in both descriptions the boundary state and the open string vertices and show that they give rise to different string amplitudes. In particular, the description based on the gauge bundle has open string vertex operators with momentum dependent Chan-Paton factors.

Keywords: M-Theory, Supersymmetry Breaking, Intersecting branes models.

*Work partially supported by the European Community's Human Potential Programme under contract MRTN-CT-2004-005104 "Constituents, Fundamental Forces and Symmetries of the Universe" and by the Italian M.I.U.R. under contract PRIN-2005023102 "Strings, D-branes and Gauge Theories". Three of us (PDV, RM, FP) thank the Galileo Galilei Institute for Theoretical Physics for their kind hospitality and for partial support during the completion of this work.

JHEP11(2007)100

Contents

1. Introduction

String theories are described by two-dimensional conformal field theories and are consistent only in twenty-six (bosonic string) or ten (superstring) space-time dimensions. As it is well-known, phenomenology imposes all of them to be compactified, except the four dimensions observed in our universe. The compactification procedure, however, always introduces a certain number of new fields called *moduli* whose expectation values are related, for example, to the size and the shape of the compact manifold and determine the parameters of the four dimensional effective Lagrangian. Without fixing these expectation values, string models are not predictive. A lot of work has been done in the last few years to fix the value of these moduli, but we are left with the problem that there are too many possibilities and there is the feeling among many string theorists that something important is still missing.

In general, in order to obtain an $\mathcal{N} = 1$ supersymmetric version of the Standard Model, one needs to compactify the six extra dimensions in a Calabi-Yau six-dimensional space. However, it is not possible, in general, to have an explicit formulation of string theory in these backgrounds. Therefore it is not really possible to construct explicit extensions of the Standard Model and compare their results with phenomenology. If one wants to do that, then it is necessary to restrict oneself to orbifolds and orientifolds of toroidal compactifications.

Starting from the observation that one wants chiral fermions, as required by the Standard Model, string theory models based on intersecting branes have been proposed and extensively studied. In particular, type IIA orientifolds with intersecting D6 branes, together with their counterparts in type IIB theory, have provided a phenomenologically interesting class of very explicit string compactifications [1–4].

In these models one only considers the simplest case in which the six-dimensional compact manifold is the product of three two-dimensional tori $T^2 \times T^2 \times T^2$. If, for the sake of simplicity, the analysis is limited to the torus described by the two coordinates x^1 and x^2 with one stack of D6 branes being placed along the x^1 axis and a different one at an angle θ $(0 \leq \theta \leq \pi)$ in the plane (x^1, x^2), then one gets the following boundary conditions for an open string having one endpoint attached to the first stack of branes and the other endpoint attached to the other stack:

$$\partial_\sigma \left[\cos(\theta)X^1 - \sin(\theta)X^2\right]_{\sigma=0} = 0 \quad ; \quad \partial_\tau \left[\sin(\theta)X^1 + \cos(\theta)X^2\right]_{\sigma=0} = 0 \tag{1.1}$$

and Neumann and Dirichlet boundary conditions respectively in the $1, 2$ directions at $\sigma = \pi$.

If one considers a squared torus with radii R_1 and R_2, then the angle θ is easily seen to be given by:

$$\tan\theta = \frac{mR_2}{nR_1}$$

where (n, m) are the wrapping numbers along the directions 1 and 2 respectively of the second stack of branes. If one performs a T-duality transformation along the direction x^2, that amounts to exchange $\sigma \leftrightarrow \tau$, the boundary conditions in eq. (1.1) are transformed into the following ones:

$$\left[\partial_\sigma X^1 - \tan\pi\nu\partial_\tau X^2\right]_{\sigma=0} = 0 \quad ; \quad \left[\partial_\sigma X^2 + \tan\pi\nu\partial_\tau X^1\right]_{\sigma=0} = 0$$

that are the boundary conditions for an open string having the endpoint at $\sigma = 0$ attached to a brane with a constant magnetic field, given by:

$$\tan\pi\nu \equiv 2\pi\alpha' f_{12} = \frac{m}{n}\frac{\alpha'}{R_1R_2},$$

JHEP11(2007)100

where $\tan \pi\nu$ is obtained from $\tan\theta$ by the T-duality transformation: $R_2 \to \frac{\alpha'}{R_2}$. In the T-dual theory the integer n multiplies the volume of the T-dual torus, and thus it plays the role of the wrapping number of the brane on the whole torus.[1]

Being the brane compactified on a torus T^2, the first Chern class for an SU(N) non-abelian gauge field living on it must be an integer, i.e.:

$$\int_{\mathcal{M}} Tr\left(\frac{F}{2\pi}\right) = 2\pi\alpha' F_{12} N = m' \Longrightarrow 2\pi\alpha' F_{12} = \frac{m'}{N} \tag{1.2}$$

being $\mathcal{M}$ the brane worldvolume. It is easily seen that the wrapping number m, along the direction x^2 in which the T-duality transformation is performed, becomes the magnetic charge m' and the wrapping number n along the other direction x^1 becomes the rank of the gauge group N that is also equal to the wrapping number on the entire torus T^2. In conclusion, this analysis suggests that a brane wrapped n times along the first cycle of the torus and m times along the second cycle of the torus becomes, under a T-duality performed along the second axis x^2, a magnetized brane with magnetic flux m that is wrapped n times along the whole torus and that is described by a non-abelian gauge theory with gauge group U(N) with $N = n$.

The fact that, in order to describe branes wrapped N times on the torus T^2, one needs a non-abelian gauge theory U(N), has been advocated by many authors [5–9]. In particular, in ref. [10], it has been used for computing, among other things, the Yukawa couplings in the field theory limit ($\alpha' \to 0$) corresponding to massless open string states attached to magnetized branes. In this approach a D-brane wrapped N times along the torus T^2 is described by a U(N) gauge theory just like a stack of N D-branes only wrapped once. The difference between the two systems is that in the latter case the gauge holonomy is the identity, while it is not trivial in the former. This implies that for N branes wrapped once the gauge theory quantities are periodic in going around the two cycles of the torus, while for an N-tuply wound D brane one gets instead a gauge bundle whose non gauge invariant quantities are periodic only up to a gauge transformation.

At this point a question is natural: do the previous considerations mean that a brane wrapped N times around the torus T^2 is necessarily described by a gauge bundle of a U(N) gauge theory? Instead of trying to answer this question directly, let us observe that, if one uses an abelian rather than a non-abelian field, then the factor N in eq. (1.2) could be in principle reproduced in a different way. In fact, in this case, eq. (1.2) becomes:

$$\int_{\mathcal{M}} \left(\frac{F}{2\pi}\right) = 2\pi\alpha' F_{12} N = m' \tag{1.3}$$

where now the factor $N = n$ is not obtained from the trace over the non-abelian group as before, but from the fact that the brane worldvolume is N times the volume of the torus $(2\pi\sqrt{\alpha'})^2$, differently from the non-abelian case where the brane worldvolume coincides

[1] This wrapping number, which we use in the paper, is not the same thing of the geometrical embedding since it contains less information. A geometrical embedding of a T^2 into T^2 is characterized by a matrix $\begin{pmatrix} p & j \\ 0 & q \end{pmatrix}$, up to SL$(2, Z)$ transformations, which has wrapping $n = pq$.

with the torus, as a consequence of the different periodicity conditions on the embedding coordinates.

This second logical possibility can be realized, for instance, by a particular system of D-branes that we call *Narain branes*. These are indeed obtained from a system of non-magnetized branes by acting with the Narain T-duality group. It seems to us that this point of view has been taken in refs. [11 − 16]. This was also the point of view taken in refs. [17, 18], but with some differences with respect to the previous ones.

In this paper we will discuss both points of views and compare them. There is also another reason for our analysis. In fact, while on the side of the intersecting branes a unique and complete string description (up to non geometrical data) is available and amplitudes involving both twisted and untwisted open strings have been computed, on the side of magnetized branes, instead, only partial tree-level string calculations have been performed and, as far as we can see, a complete string description of magnetized branes is still lacking. For example, in the case of the Yukawa couplings, only a part of the amplitude has been computed at tree-level in string theory [4, 19], while the rest has been obtained in the field theory limit [10]. The complete expression of the Yukawa couplings has been obtained from the corresponding computations in the intersecting branes scenario via T-duality [20 − 22] or from the two-loop twisted partition function [23, 24].

In this paper we make the first step toward a more complete string theory formulation of magnetized branes on the torus.

In particular, on the one hand, we show that, in order to describe wrapped branes, we need to extend the concept of gauge bundle to string theory and in this framework we write the equations that characterize the physical states when one goes around the two one-cycles of the torus. We show that in this case the Chan-Paton factors, unlike the usual ones describing the non-abelian degrees of freedom, are momentum dependent. Then we construct the boundary state corresponding to wrapped magnetized branes described by gauge bundles and compute the one-loop partition function. The boundary state is constructed in two different ways. The first consists in factorizing the annulus diagram computed in the open string channel which fixes the boundary state up to a phase factor and the second in a direct calculation involving the non-abelian Wilson loop. We find agreement between the two procedures up to a phase factor.

On the other hand, following the other logical possibility discussed above for describing the wrapped branes, we start from a theory with no gauge field and we get a theory with a non-vanishing gauge field by using the general T-duality group found by Narain. We then apply the same technique based on T-duality for constructing a boundary state with a gauge field on it from the usual boundary state. It turns out that the boundary states determined in the two approaches are not identical, but give the same one-loop amplitude. The boundary state obtained in the gauge bundle scheme contains an extra phase factor that, however, does not contribute to the annulus diagram.

We then compute in these two theories disk amplitudes involving both open and closed strings showing that they are different. Hence, if we only focus on the first Chern class constraint in order to characterize the T-dual of intersecting branes, we do not have yet enough elements for distinguishing which of these two inequivalent descriptions is the right one.

JHEP11(2007)100

The paper is organized as follows. In section 2 we study open strings in an arbitrary toroidal background interacting with a magnetic field living on the compactified directions. We introduce the conserved generalized translation operator and the notion of gauge bundle.

In section 3 we discuss the gauge bundle in string theory as a description of wrapped magnetized space-filling branes. We construct the corresponding boundary state, compute the one-loop diagram, and give the open string vertices containing the Chan-Paton factors that are momentum dependent. In this paper we will call *non-abelian branes* those based on a non-abelian gauge bundle.

In section 4 we discuss the Narain branes. In particular, we construct their boundary state, that turns out to be equal to that of the non-abelian branes apart from a phase factor, and their open string vertices. We show that the two kinds of branes, even if they have the same free-energy, are indeed different objects because they have a different boundary state and different scattering amplitudes involving both open and closed strings.

Many of the calculations are presented in four appendices. In appendix A we summarize our conventions. Appendix B is devoted to the solution of the equations of motion of open and closed strings in closed and open string backgrounds. In appendix C we discuss the transformations of various quantities under the general Narain group of T-duality and in appendix D we perform the path-ordering calculation of the boundary state with a background gauge field.

2. Open strings in flux backgrounds

In this section we study the effects of turning on a background gauge field living on a $\hat{d}$-dimensional torus and interacting with closed string backgrounds. First we review the solution of the equations of motion for open strings (the case of closed strings is discussed in appendix B) and then we analyse how the translation generator gets modified when a magnetic field is turned on.

2.1 Open string in open and closed string background.

Let us consider open strings on the $\hat{d}$-dimensional torus, interacting with constant gravitational and Kalb-Ramond backgrounds and with an open string background consisting of two abelian gauge fields with constant field strengths $F^{(0)}_{ij}$ and $F^{(\pi)}_{ij}$ acting at the two end-points $\sigma = 0, \pi$ of the string and, in general, independent of each other. Such a system is described by the following action:[2]

$$S = S_{\text{bulk}} + S_{\text{boundary}} \tag{2.1}$$

where S_{bulk} is given by:

$$S_{\text{bulk}} = -\frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi d\sigma \left[G_{ab}\partial_\alpha X^a \partial_\beta X^b \eta^{\alpha\beta} - B_{ab}\epsilon^{\alpha\beta}\partial_\alpha X^a \partial_\beta X^b\right] \tag{2.2}$$

[2]With respect to the notation used in [18] we have set $q_\pi \to -q_\pi$.

being the world-sheet metric $\eta_{\alpha\beta} = \mathrm{diag}(-1,1)$ and $\epsilon^{01} = 1$, while S_{boundary} is:

$$\begin{aligned} S_{\mathrm{boundary}} &= -q_0 \int d\tau A_i^{(0)} \partial_\tau X^i|_{\sigma=0} + q_\pi \int d\tau A_i^{(\pi)} \partial_\tau X^i|_{\sigma=\pi} \\ &= \frac{q_0}{2} \int d\tau F_{ij}^{(0)} X^j \dot{X}^i|_{\sigma=0} - \frac{q_\pi}{2} \int d\tau F_{ij}^{(\pi)} X^j \dot{X}^i|_{\sigma=\pi} \quad ; \quad i,j = 1,\ldots,\hat{d} \end{aligned} \tag{2.3}$$

where q_0 and q_π are the charges located at the two end-points of the open string. In eq. (2.3) we have used the following form for the background gauge fields

$$A_i = -\frac{1}{2} F_{ij} x^j \ . \tag{2.4}$$

The field A_i in eq. (2.4) is not a periodic quantity when we go around one of the two one-cycles of the torus. However, on the torus, gauge non-invariant quantities as A_i have only to be periodic up to a gauge transformation [25]:

$$A_i(x^j + 2\pi\sqrt{\alpha'}\delta_l^j) = \Omega_l(x)\ A_i(x^j)\ \Omega_l^{-1}(x) - i\frac{1}{q}\Omega_l(x)\ \partial_i \Omega_l^{-1}(x) \tag{2.5}$$

where q is the gauge coupling constant and $\Omega_l(x) \equiv \Omega_l(x^{j\neq l})$ is the gauge transition function. From now we mean by *gauge bundle* the assignment of a background field, together with a transition function which fixes the periodicity property of the gauge field. Analogously, matter fields in the adjoint representation have to satisfy the periodicity conditions

$$\Phi(x^j + 2\pi\sqrt{\alpha'}\delta_l^j) = \Omega_l(x)\ \Phi(x^j)\ \Omega_l^{-1}(x). \tag{2.6}$$

Notice that, if we perform a gauge transformation

$$A_i^\omega(x) = \omega(x)\ A_i(x)\ \omega^{-1}(x) - \frac{i}{q}\omega(x)\ \partial_i\omega^{-1}(x)$$

the transition functions transform as

$$\Omega_j^\omega(x) = \omega(x^1,\ldots,x^j + 2\pi\sqrt{\alpha'},\ldots,x^{\hat{d}})\ \Omega_j(x)\ \omega^{-1}(x^1,\ldots,x^{\hat{d}}). \tag{2.7}$$

Furthermore, they also have to satisfy the *cocycle condition*, which simply means that the gauge fields must be unchanged when translated along a closed path:

$$\Omega_j(x^k + 2\pi\sqrt{\alpha'}\delta_i^k)\Omega_i(x^k)\Omega_j^{-1}(x^k)\Omega_i^{-1}(x^k + 2\pi\sqrt{\alpha'}\delta_j^k) = \mathbb{I}_N. \tag{2.8}$$

For the choice of the gauge field given in eq. (2.4), the gauge transition functions satisfying the identity in eq. (2.5) are

$$\Omega_i(x) = e^{-i\pi\sqrt{\alpha'}qF_{ij}x^j}. \tag{2.9}$$

With the previous choice the cocycle condition is trivially satisfied because it is equivalent to require that the first Chern class is an integer, as we will show shortly.

The above considerations can be extended to the case of a U(1) gauge field that is included in a U(N) gauge theory. In this case we have a non-abelian gauge bundle where the gauge transition functions are non-trivial unitary matrices. A possible choice for them is [25]:

$$\Omega_j = e^{-i\pi\sqrt{\alpha'}qF_{ji}x^i}\omega_j$$

where we have extracted the U(1) factor from the gauge transition functions. The cocycle condition imposes the following constraints on the ω's:

$$\omega_i\ \omega_j = e^{+i2\pi q\hat{F}_{ij}}\ \omega_j\ \omega_i \ \ ; \qquad \hat{F}_{ij} \equiv 2\pi\alpha' F_{ij}. \tag{2.10}$$

By taking the determinant of the previous expression it follows that the field strength must satisfy the condition:

$$q\hat{F}_{ij}\ N = n_{ij} \in \mathbb{Z} \tag{2.11}$$

where F_{ij} is constant and n_{ij} is an integer. Eq. (2.11) is indeed satisfied because it coincides with the first Chern class for a non-abelian gauge field, that must be an integer. In the case of an SU(N) gauge group, one can explicitly construct the matrices ω_i in terms of the constant 't Hooft matrices P_N and Q_N [25] given in appendix A:

$$\omega_i \equiv P_N^{s_i} Q_N^{t_i} \qquad i = 1, \ldots, \hat{d} \tag{2.12}$$

with s_i and t_i suitable integers. Since $P_N Q_N = e^{2\pi i/N} Q_N P_N$, then the cocycle condition written in eq. (2.10) can be easily satisfied by choosing:

$$n_{ij} = s_i t_j - s_j t_i.$$

with n_{ij} defined in eq. (2.11). One could also add Wilson lines and this can be done in two different ways: either by adding them as a constant in the expression of the gauge field or in the transition functions. We will discuss these two possibilities later on.

From the action in eq. (2.1) one can derive the equations of motion in the bulk given by:

$$\partial_\alpha [G_{ij} \partial^\alpha X^j] = 0 \tag{2.13}$$

and the two boundary conditions at $\sigma = 0, \pi$:

$$\Big[G_{ij}\partial_\sigma X^j + (B_{ij} - 2\pi\alpha' q_{0,\pi} F_{ij}^{(0,\pi)})\partial_\tau X^j\Big]_{\sigma=0,\pi} = 0. \tag{2.14}$$

The solution of these equations can be easily found in the simplest case in which

$$q_0 = q_\pi = q \qquad F_{ij}^{(0)} = F_{ij}^{(\pi)} = F_{ij}.$$

and the following condition holds:

$$\det(q_0 F^{(0)} - q_\pi F^{(\pi)})_{ij} = 0$$

corresponding to the so-called *dipole* string. The case in which this determinant is different from zero corresponds to the *dycharged* string.

In the dipole case the general solution [26, 27] can be written as (see appendix B.3 for details):

$$X^i(\sigma,\tau) = \frac{1}{2}\left[\hat{X}^i_L(\tau+\sigma) + \hat{X}^i_R(\tau-\sigma)\right] \tag{2.15}$$

where the left and right moving parts are given, up to a constant, by:

$$\hat{X}^i_L(\tau+\sigma) = (G^{-1}\mathcal{E})^i_j\left(X^j_L(\tau+\sigma)\right) \tag{2.16}$$

and

$$\hat{X}^i_R(\tau-\sigma) = (G^{-1}\mathcal{E}^T)^i_j\left(X^j_R(\tau-\sigma)\right) \tag{2.17}$$

with $\mathcal{E}$ defined by:

$$\mathcal{E} = \parallel \mathcal{E}_{ij} \parallel = E^T + 2\pi\alpha' q_0 F \equiv G - \mathcal{B} \tag{2.18}$$

being

$$E = \parallel E_{ij} \parallel = G + B$$

and

$$\begin{aligned} X^i_L(\tau+\sigma) &= x^i + 2\alpha'\mathcal{G}^{ij}p_j(\tau+\sigma) + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^i_n}{n}e^{-in(\tau+\sigma)} \\ X^i_R(\tau-\sigma) &= x^i + 2\alpha'\mathcal{G}^{ij}p_j(\tau-\sigma) + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^i_n}{n}e^{-in(\tau-\sigma)} \end{aligned}$$

where $\mathcal{G}_{ij}$ is the *open string metric*:

$$\mathcal{G} \equiv \mathcal{E}^T G^{-1}\mathcal{E}$$

and $\mathcal{G}^{ij}$ is its inverse. The quantization of the theory requires the following commutation relations [26 – 28]:

$$[x^i, x^j] = i\ 2\pi\alpha'\Theta^{ij} \qquad [x^i, p^j] = i\mathcal{G}^{ij} \qquad [\alpha^i_m, \alpha^j_n] = \mathcal{G}^{ij} m\delta_{n+m,0}$$

where Θ is defined by the relation:

$$(\mathcal{E}^{-1})^{ij} = \mathcal{G}^{ij} - \Theta^{ij}. \tag{2.19}$$

The operator L_0 is given by the Hamiltonian in eq. (B.7) and a straightforward calculation gives:

$$L_0 = \alpha' p_i \mathcal{G}^{ij} p_j + \sum_{n=1}^{\infty}\mathcal{G}_{ij}\alpha^i_{-n}\alpha^j_n. \tag{2.20}$$

2.2 Translation generator in the presence of a magnetic field.

In our compactified string theory, described by the bulk and boundary actions respectively given in eqs. (2.2) and (2.3), the conjugate momentum density given by:

$$P_i = \frac{1}{2\pi\alpha'}\left[G_{ij}\dot{X}^j + B_{ij}X'^j\right]$$
$$+\frac{1}{2}q_0 F^{(0)}_{ij}X^j(0)\;\delta(\sigma) - \frac{1}{2}q_\pi F^{(\pi)}_{ij}X^j(\pi)\;\delta(\pi-\sigma)$$

is not gauge invariant, because of the gauge choice made in eq. (2.4) and is not a conserved charge as in the case with $F = 0$. This is not surprising because the string action, in the presence of a magnetic field, is not invariant under translations. However, one can easily see that the string action is indeed invariant under a suitable combination of a translation and a gauge transformation [29] . In particular, with the gauge chosen in eq. (2.4), the action is invariant under the combination of a translation $X^i \to X^i + \epsilon^i$ (under which the gauge field transforms as $A_i \to A_i + \partial_j A_i \epsilon^j$) and a gauge transformation $A_i \to A_i - \partial_i\phi$ with $\phi = \frac{1}{2}F_{ij}X^j\epsilon^i$. We will refer to this transformation as a *generalized translation*. The Nöther current associated to such an invariance is given by:

$$J^\alpha_i(\tau,\sigma) = -\frac{1}{2\pi\alpha'}[G_{ij}\partial^\alpha X^j - B_{ij}\epsilon^{\alpha\beta}\partial_\beta X^j]$$

which is conserved as a consequence of the equations of motion.

The conservation of the previous current implies [30]:

$$\begin{aligned}
0 &= \int_0^\pi d\sigma\partial_\alpha J^\alpha_i \\
&= \int_0^\pi d\sigma\,\partial_\tau J^0_i(\sigma) + \left[J^1_i(\sigma)|_{\sigma=\pi} - J^1_i(\sigma)|_{\sigma=0}\right] \\
&= \partial_\tau\left[\int_0^\pi d\sigma\frac{1}{2\pi\alpha'}[G_{ij}\dot{X}^j + B_{ij}X^{j\prime}] + q_0F^{(0)}_{ij}X^j|_{\sigma=0} - q_\pi F^{(\pi)}_{ij}X^j|_{\sigma=\pi}\right]
\end{aligned}$$

where we have used the open string boundary conditions given in eqs. (2.14). It follows that the generator of such a transformation is a constant of the motion that is simply given by [29, 30]:

$$\begin{aligned}
\hat{T}_i &= \int_0^\pi d\sigma\left\{\frac{1}{2\pi\alpha'}[G_{ij}\,\dot{X}^j + B_{ij}\,X^{j\prime}] + q_0F^{(0)}_{ij}\,X^j\delta(\sigma) - q_\pi F^{(\pi)}_{ij}\,X^j\delta(\sigma-\pi)\right\} \\
&= (q_0F^{(0)} - q_\pi F^{(\pi)})_{ij}x^j + \delta_{q_0F^{(0)}-q_\pi F^{(\pi)};0}\,\mathcal{G}_{ij}p^j. \qquad (2.21)
\end{aligned}$$

It differs from the conjugate momentum for the one half factor in front of the terms depending on the gauge field. Moreover it satisfies the following commutation relation [30]:

$$\left[\hat{T}_i,\hat{T}_j\right] = i\left(q_0F^{(0)} - q_\pi F^{(\pi)}\right)_{ij}. \qquad (2.22)$$

It is interesting to observe that, when the dipole condition is satisfied, the same expression for the translation operator could be obtained by considering a slight modification of eq. (2.3):

$$\hat{S}_{\text{boundary}} = -q\int d\tau\int_0^\pi d\sigma\;F_{ij}X'^j\dot{X}^i$$

$$= -q \int d\tau \int_0^\pi d\sigma \left[\partial_\sigma \left(\frac{1}{2} F_{ij} X^j \dot{X}^i \right) - \partial_\tau \left(\frac{1}{2} q F_{ij} X^j X'^i \right) \right]. \tag{2.23}$$

This expression is equivalent to eq. (2.3) up to a total derivative with respect to τ and is gauge invariant. The conjugate momentum computed from the bulk action in eq. (2.2) with the addition of the boundary term in eq. (2.23) turns out to be:

$$\begin{aligned}
\hat{P}_i &= \frac{1}{2\pi\alpha'} \left[G_{ij} \dot{X}^j + B_{ij} X'^j \right] - q F_{ij} X'^j \\
&= \frac{1}{2\pi\alpha'} \left[G_{ij} \dot{X}^j + (B_{ij} - q\hat{F}_{ij}) X'^j \right] \\
&= \frac{1}{2\pi\alpha'} \left[(\mathcal{E}^T)_{ij} \partial_+ X^j + (\mathcal{E})_{ij} \partial_- X^j \right].
\end{aligned} \tag{2.24}$$

When integrated in $d\sigma$, it yields precisely eq. (2.21) for the dipole case.

On a torus the system must be invariant under a generalized discrete translation, i.e. $x^i \to x^i + 2\pi\sqrt{\alpha'}$. This means that the physical states are those which satisfy the following identity:

$$\mathcal{T}_i |phys\rangle \equiv e^{i 2\pi \sqrt{\alpha'} \hat{T}_i} |phys\rangle = |phys\rangle. \tag{2.25}$$

In particular, when the dipole condition is satisfied, the translation operator becomes identical to the momentum operator in eq. (2.24) (as it follows from eq. (2.21)) whose spectrum is determined by imposing the previous constraint. In this way one gets:

$$p_i |k\rangle = k_i |k\rangle = \frac{n_i}{\sqrt{\alpha'}} |k\rangle$$

where k_i is the eigenvalue of the operator p_i. Taking into account these eigenvalues one can rewrite the operator L_0 in eq. (2.20) as follows:

$$L_0 = n_i \mathcal{G}^{ij} n_j + \sum_{n=1}^{\infty} \mathcal{G}_{ij} \alpha^i_{-n} \alpha^j_n. \tag{2.26}$$

The previous analysis is valid when we have just D-branes wrapped once on the torus and having a U(1) background gauge field turned-on on their worldvolume. In the next section we will generalize the previous construction to the case of D-branes wrapped N times.

3. Non-abelian branes

In this section we provide a description of D-branes wrapped N times along the two-cycles of the torus T^2. We start discussing in section 3.1 the case with vanishing gauge field that has been already discussed in the literature and then we analyze in section 3.2 the case with $F \neq 0$. We show that, in this description, D-branes wrapped N times along the torus T^2 have Chan-Paton factors that are momentum dependent. Then in section 3.3 we discuss the non-degenerate case and in section 3.4 the degenerate one. section 3.5 is devoted to the construction of the boundary state and to the one-loop annulus diagram. Finally the open tachyon vertex is constructed in section 3.6.

JHEP11(2007)100

3.1 Non-abelian gauge bundle: $F = 0$

Let us start discussing the case with $F = 0$, but with A having a non trivial background value or, equivalently, when a Wilson line background is turned on. This is the case considered in ref. [8] where the perturbative dynamics of open strings attached to multiply wound D-branes is analyzed. A very important point made in ref. [8] is that a D-brane wrapped N times around a torus can be described as a brane with gauge group $\mathrm{U}(N)$ and with a non trivial holonomy group $\mathcal{H} \neq \mathbb{I}$. Actually, as stressed in ref. [31], the existence of a non-trivial holonomy is what makes the difference between a bound state of N D-branes each wrapped once around the torus (such a state has gauge group $\mathrm{U}(N)$ but trivial holonomy $\mathcal{H} = \mathbb{I}$) and a single N-tuply wound D-brane. On the other hand, a non-trivial gauge holonomy $\mathcal{H}$ arises when a Wilson line background is turned on, being $\mathcal{H} = P[e^{+iq\oint A}]$ and P stands for the path ordering. Thus Wilson lines provide a natural way to describe multiply wound D-branes. However, as discussed in ref. [8], a Wilson line background implies non standard kinetic terms for the fields. Therefore it is useful to make a field redefinition which actually exchanges the Wilson line background with non-trivial boundary conditions. More explicitly, let us consider the following $\mathrm{U}(N)$ Wilson lines background:

$$A_i = \Theta_i = \begin{pmatrix} a_i^1 & \dots & 0 \\ 0 & \ddots & \\ 0 & \dots & a_i^N \end{pmatrix}.$$

It implies the existence of a non-trivial holonomy group:

$$\mathcal{H} = P[e^{iq\oint A}] = e^{iq2\pi\Theta_i\sqrt{\alpha'}}. \tag{3.1}$$

Being the gauge field background A_i constant, the periodicity condition in eq. (2.5) is simply satisfied by the gauge transition function $\Omega_l = \mathbb{I}_N$, which means that both the gauge and the matter fields have trivial boundary conditions:

$$A_i(x^j + 2\pi\sqrt{\alpha'}\delta_l^j) = A_i(x^j) \qquad \Phi(x^j + 2\pi\sqrt{\alpha'}\delta_l^j) = \Phi(x^j). \tag{3.2}$$

However, the non-zero value of the background field A generates non-standard kinetic terms. It is therefore useful to perform a gauge transformation with the gauge function $\omega(x^1, \dots, x^{\hat{d}}) = e^{iq\Theta_i x^i}$ so, according to eq. (2.7), one gets:

$$A_i^\omega = 0 \qquad \Omega_i^\omega = e^{iq2\pi\Theta_i\sqrt{\alpha'}} \tag{3.3}$$

corresponding to a new description in which the Wilson line background is zero, but the gauge and matter fields satisfy the non-trivial boundary conditions given in eqs. (2.5), (2.6) with Ω given in (3.3). Notice that the transition function in eq. (3.3) coincides with the holonomy matrix in eq. (3.1).

3.2 Non-abelian gauge bundle in string theory: $F \neq 0$

In this subsection we discuss the case $F \neq 0$ and extend the notion of gauge bundle, discussed in section 2.1, to the string level in the non-abelian case. The main difference

JHEP11(2007)100

which occurs in the $F \neq 0$ case is that, by choosing for example the gauge field as in eq. (2.4), the gauge transition functions are forced to be non trivial, because A_i itself is not single valued under $x^i \to x^i + 2\pi\sqrt{\alpha'}$. In other words with $F \neq 0$ one is forced to have non trivial holonomy.

Let us first go back to the abelian case describing a D brane only wrapped once on the torus T^2. The basic ingredient is the identification of the physical states under combined translations and gauge transformations as expressed in eq. (2.25). In addition we have also to impose the consistency condition which requires the string states to be left invariant when translated along a closed path. This means that:

$$\mathcal{T}^{(i)}_{2\pi\sqrt{\alpha'}}\mathcal{T}^{(j)}_{2\pi\sqrt{\alpha'}}|phys.\rangle = \mathcal{T}^{(j)}_{2\pi\sqrt{\alpha'}}\mathcal{T}^{(i)}_{2\pi\sqrt{\alpha'}}|phys.\rangle \tag{3.4}$$

or equivalently:

$$\left[\mathcal{T}^{(i)}_{2\pi\sqrt{\alpha'}},\, \mathcal{T}^{(j)}_{2\pi\sqrt{\alpha'}}\right] = 0. \tag{3.5}$$

Eq. (3.5) can be considered as the extension to the string level of the cocycle condition given in eq. (2.8). In the abelian case the constraint in eq. (3.5) is always verified. In particular, when the dipole condition holds, it is satisfied because the translation generators commute, as it follows from eq. (2.22). In the dycharged string case, the right hand side of eq. (2.22) is not vanishing anymore and eq. (3.5) imposes the following constraint:

$$(2\pi\sqrt{\alpha'})^2(q_0F^{(0)} - q_\pi F^{(\pi)})_{ij} = 2\pi(n^{(0)}_{ij} - n^{(\pi)}_{ij}) \quad n_{ij} \in \mathbb{Z} \tag{3.6}$$

which is indeed satisfied because this is nothing but the definition of the first Chern class for a constant field strength. This is another evidence that $\mathcal{T}$ is the right translation operator.

The previous analysis is valid in the abelian case in which we have just two branes wrapped once on the torus with two different $\mathrm{U}(1)$ background gauge fields turned-on on their worldvolume. But if we want to describe two stacks of D-branes wrapped respectively N_0 and N_π times on the torus, we need to extend the previous considerations to a non-abelian case where the string states are dressed with Chan-Paton factors. The gauge group is $\mathrm{U}(N_0) \times \mathrm{U}(N_\pi)$. In this case the generator of generalized translations acts also on the Chan-Paton factors. We denote by $\omega_i^{(0,\pi)}$ the matrices acting on them. Physical states must then satisfy the identification

$$e^{2\pi\sqrt{\alpha'}\,i\hat{T}_i}(\omega_i^{(0)\dagger})_{\ell h}(\omega_i^{(\pi)})_{tm}|\Phi,\, h\, t\rangle \equiv |\Phi,\, \ell\, m\rangle, \tag{3.7}$$

where the background gauge field has been taken in the $\mathrm{U}(1)$ part of the gauge group. Moreover the following cocycle condition has to be satisfied:

$$\begin{aligned} &e^{(2\pi\sqrt{\alpha'})i\hat{T}_j}e^{(2\pi\sqrt{\alpha'})i\hat{T}_i}(\omega_i^{(0)}\,\omega_j^{(0)})^\dagger_{\ell h}(\omega_i^{(\pi)}\,\omega_j^{(\pi)})_{tm}|\Phi,\, h\, t\rangle \\ &\qquad = e^{(2\pi\sqrt{\alpha'})i\hat{T}_i}e^{(2\pi\sqrt{\alpha'})i\hat{T}_j}(\omega_j^{(0)}\,\omega_i^{(0)})^\dagger_{\ell h}(\omega_j^{(\pi)}\,\omega_i^{(\pi)})_{tm}|\Phi,\, h\, t\rangle. \end{aligned} \tag{3.8}$$

This equation is satisfied if we impose the relations:

$$\omega_i^{(0)}\,\omega_j^{(0)} = e^{+2\pi i\frac{n^{(0)}_{ij}}{N_0}}\omega_j^{(0)}\,\omega_i^{(0)} \quad ; \quad \omega_i^{(\pi)}\,\omega_j^{(\pi)} = e^{+2\pi i\frac{n^{(\pi)}_{ij}}{N_\pi}}\omega_j^{(\pi)}\,\omega_i^{(\pi)}. \tag{3.9}$$

Eq. (3.9) is the string realization of the constraint given in eq. (2.10) and again it can be satisfied by taking for the ω's the matrices in eq. (2.12).

Eq. (3.7) can also be written in the following suggestive form:

$$e^{2\pi\sqrt{\alpha'}\, i\hat{T}_i}[(\omega_i^{(0)})^\dagger \Lambda(\omega_i^{(\pi)})]_{\ell m} \lim_{z\to 0} V(z)|0\rangle \equiv \Lambda_{\ell m} \lim_{z\to 0} V(z)|0\rangle$$

where $V(z)$ is the vertex operator which creates the corresponding physical state by acting on the conformal vacuum and the matrix Λ is the Chan-Paton factor. In the dipole case where $q_0 F^{(0)} = q_\pi F^{(\pi)}$ implies $[\hat{T}_i, \hat{T}_j] = 0$ so that we can assume that the generators of generalized translation annihilate the vacuum, i.e. $\hat{T}_i|0\rangle = 0$, the previous equation implies the important relation:

$$e^{2\pi\sqrt{\alpha'}\, i\hat{T}_i}\omega_i^{(0)} \Lambda V(z) (e^{2\pi\sqrt{\alpha'}\, i\hat{T}_i}\omega_i^{(\pi)})^\dagger \equiv \Lambda V(z) \tag{3.10}$$

which must be satisfied by the open string vertex $V(z)$.

In the last part of this section we will analyze more explicitly the case of a constant background gauge field satisfying the dipole condition and living in the identity part of the U(N) gauge group. It follows that the first Chern class is given by:

$$\int \mathrm{Tr}\left(\frac{q\, F_{12}\mathbb{I}_N}{2\pi}\right) = 2\pi\alpha' q\, F_{12} N = n_{12} \equiv f \in \mathbb{Z} \tag{3.11}$$

where we have used the fact that $\int d^2x = (2\pi\sqrt{\alpha'})^2$. In this set-up the cocycle condition given in eq. (3.9) can be easily satisfied by taking:

$$\omega_1 = Q_N \qquad \omega_2 = P_N^{-f}. \tag{3.12}$$

In general, the Chan-Paton factor Λ depends on the choice of transition functions. With the previous choice, eq. (3.7) gives the following constraints:

$$\begin{aligned} e^{2\pi\sqrt{\alpha'}\, ip_1}\, Q_N^\dagger\, \Lambda_k\, Q_N |\Phi,\, k\rangle &\equiv \Lambda_k |\Phi,\, k\rangle \\ e^{2\pi\sqrt{\alpha'}\, ip_2}\, P_N^f\, \Lambda_k\, P_N^{-f} |\Phi,\, k\rangle &\equiv \Lambda_k |\Phi,\, k\rangle \end{aligned} \tag{3.13}$$

where we have denoted by k the momentum of the physical state and, as it will be clear later, we have also assumed a dependence of the Chan-Paton factors on the momentum k.

It is useful to expand the Chan-Paton factors in terms of the complete set of 't Hooft matrices:

$$(\Lambda_k)_{lm} = \sum_{h_1,\, h_2=0}^{N-1} C_{(k,\, h_i)} \left(Q_N^{h_2} P_N^{h_1}\right)_{lm}. \tag{3.14}$$

We will see in the next subsections that eqs. (3.13) fix the structure of the Chan-Paton factors up to a c-number. When discussing the string vertices, we will explicitly show that a phase factor is indeed necessary to ensure the correct hermitian conjugation property of the string vertex.

In the following we are going to treat separately the degenerate case and the non-degenerate one, where the great common divisor (GCD) between the first Chern class and the rank of the gauge group is greater or equal to one, respectively.

JHEP11(2007)100

3.3 The non-degenerate case: $GCD(f, N) = 1$

In this case, by translating of $2\pi\sqrt{\alpha'}$ the string states N times along the i-th direction of the torus, eqs. (3.13) reduce to the identity:

$$e^{2\pi\sqrt{\alpha'}\, i\, N\, p_i}\, \Lambda_k\, |\Phi,\, k\rangle \equiv \Lambda_k |\Phi,\, k\rangle$$

where we have used $P_N^N = Q_N^N = 1$, giving the following quantization of the momenta:

$$k_i = \frac{n_i}{\sqrt{\alpha'}\, N} \qquad n_i \in \mathbb{Z} \ \text{ and } \ i = 1,\, 2. \tag{3.15}$$

Eq. (3.15), when used in the first of eqs. (3.13), yields the following constraint on the Chan-Paton factors in eq. (3.14):

$$\begin{aligned} e^{2i\pi n_1/N} Q_N \Lambda_{(n_1,\, n_2)} Q_N^{-1} &= e^{2i\pi n_1/N} \sum_{h_1,\, h_2=0}^{N-1} e^{2i\pi h_1/N} C_{(n_1,\, n_2,\, h_1,\, h_2)} Q_N^{h_2}\, P_N^{h_1} \\ &\equiv \sum_{h_1,\, h_2=0}^{N-1} C_{(n_1,\, n_2,\, h_1,\, h_2)} Q_N^{h_2}\, P_N^{h_1} \end{aligned} \tag{3.16}$$

after having used eq. (A.1) h_1 times in the first equality. Eq. (3.16) implies

$$C_{(n_1, n_2, h_1, h_2)} = \delta^{[N]}_{-n_1, h_1} C_{(n_1, n_2)}.$$

By using instead eq. (3.13) along the direction x_2 one gets:

$$\begin{aligned} e^{2i\pi n_2/N} P_N^f \Lambda_{(n_1, n_2)} P_N^{-f} &= e^{2i\pi n_2/N} \sum_{h_1,\, h_2=0}^{N-1} e^{2i\pi\, f\, h_2/N} C_{(n_1, n_2, h_1,\, h_2)} Q_N^{h_2}\, P_N^{h_1} \\ &\equiv \sum_{h_1,\, h_2=0}^{N-1} C_{(n_1, n_2, h_1,\, h_2)} Q_N^{h_2}\, P_N^{h_1} \end{aligned} \tag{3.17}$$

which implies that all the quantities $C_{(n_i, h_i)}$ vanish unless

$$f h_2 \equiv -n_2 \quad \mathit{mod}\ N. \tag{3.18}$$

But, since $Q_N^N = 1$, h_2 is actually defined modulo N. Hence we can solve eq. (3.18) as

$$h_2 = \hat{h}_2 n_2$$

where we have defined the constant $\hat{h}_2$ as

$$f \hat{h}_2 \equiv -1 \ \ \mathit{mod}\ N \qquad 0 \le \hat{h}_2 < N \tag{3.19}$$

in such a way that for any n_2 there is only one value of h_2 (modulo N). In conclusion, the periodicity in x^1 and x^2 implies for C the following form:

$$C_{(n_1, n_2, h_1, h_2)} = \delta^{[N]}_{h_1, -n_1} \delta^{[N]}_{h_2, \hat{h}_2 n_2} C_{(n_1, n_2)} \tag{3.20}$$

Furthermore, eq. (3.20) implies that, for any value of n_1 and n_2, we have just one value of h_1 and one of $\hat{h}_2$ modulo N that allow to satisfy the periodicity in the two directions x^1 and x^2. This means that, for each value of the two integers (n_1, n_2) we have a definite value for the integers (h_1, h_2) modulo N and this selects therefore a unique matrix $(Q_N^{h_2} P_N^{h_1})_{ab}$ in the expansion in eq. (3.14). Thus the Chan Paton factors explicitly depend on the momentum.

3.4 The degenerate case: $g = GCD(f, N) > 1$

In this case we have that the periodicity conditions $\omega_i^N = 1$ are modified as follows:

$$(\omega_1)^N = (\omega_2)^{\frac{N}{g}} = 1$$

and therefore the momenta become:

$$k_1 = \frac{n_1}{\sqrt{\alpha'} N} \qquad k_2 = \frac{n_2 g}{\sqrt{\alpha'} N}.$$

By repeating the same procedure as in the non-degenerate case, we have that the condition written in eq. (3.16) is unchanged, while eq. (3.17) is modified as follows:

$$\begin{aligned} e^{2i\pi n_2 g/N}\, P_N^f \Lambda_{(n_1,n_2)} P_W^{-f} &= e^{2i\pi n_2 g/N} \sum_{h_1,h_2=0}^{N-1} C_{(n_1,n_2,h_1,h_2)} e^{2i\pi f h_2/N} Q_N^{h_2}\, P_N^{h_1} \\ &\equiv \sum_{h_1,h_2=0}^{N-1} C_{(n_1,n_2,h_1,h_2,h_i)} Q_N^{h_2}\, P_N^{h_1}. \end{aligned}$$

The following condition is implied:

$$\frac{f}{g} h_2 \equiv -n_2 \quad mod\ \frac{N}{g}. \tag{3.21}$$

The solution of this equation can be found by writing again:

$$h_2 = \hat{h}_2\, n_2 \qquad \text{with } 0 \leq \hat{h}_2 < N/g \tag{3.22}$$

and solving the following condition which is independent on n_2:

$$\frac{f}{g} \hat{h}_2 \equiv -1 \quad mod\ \frac{N}{g}.$$

Finally, we can write:

$$\Lambda_{(n_1,n_2)} \equiv C_{(n_1,\, n_2, n_1, \hat{h}_2\, n_2)} Q_N^{\hat{h}_2\, n_2 + m\frac{N}{g}}\, P_N^{-n_1} \tag{3.23}$$

with $m \in \mathbb{Z}$. However, eq. (3.23) immediately shows that we are losing some solutions. This is because, due to the periodicity of the matrix Q_N which is N and not N/g, in varying m in the interval $0 \leq m < g$ we have inequivalent solutions associated to the same momentum $n_2/\sqrt{\alpha'}N$. This is an extra degeneracy, not present in the non-degenerate case, which leads us to write, instead of eqs. (3.22) and (3.23), the following most general solutions:

$$h_2 = \hat{h}_2 n_2 + A\frac{N}{g} \qquad 0 \leq A < g$$

and

$$C_{(n_1,\hat{n}_2,h_1,h_2)} = \delta^{[N]}_{h_1,-n_1} \delta^{[N]}_{h_2,\hat{h}_2 \frac{n_2}{g} + AN/g} C_{(n_1,\, n_2,\, A)} \quad ; \quad 0 \leq A < g.$$

Hence, given n_1, we have only one value of $h_1 = n_1$ contributing in the expansion in eq. (3.14), while, given n_2, we have g possible values of h_2. This means that each value of momentum has a degeneracy g.

In the last part of this section we make some comments on the generalization of the previous bundle construction to the case of magnetized branes living on a product of $T^2 \times T^2 \cdots \times T^2$ of $\frac{\hat{d}}{2}$ tori. The gauge bundle is again $\mathrm{U}(N)$, but now it is broken into the product $\prod_{l=1}^{\hat{d}/2} \mathrm{U}(N_l)$ by the presence, in each factorized torus, of a background gauge field with constant field strength $F_{12}^{(l)}$, with $l = 1, \ldots, \hat{d}/2$. The cocycle conditions, given in eq. (3.9), can indeed be satisfied by embedding, as in the T^2 case, the background gauge fields in the abelian parts of the gauge transition functions and choosing for the non-abelian part the product of $\hat{d}/2$ constant matrices, all equal to the ones given in eq. (3.12). This choice of gauge bundle allows us to trivially generalize the previous analysis to the product of $T^2 \times T^2 \ldots T^2$ by simply adopting in each T^2 the procedure developed before. An interesting open question is to understand how general this choice is. It seems indeed to be consistent when the second Chern numbers are integers [25]. However it would be nice to find general rules in constructing consistent gauge bundles associated to branes compactified on a generic torus $T^{\hat{d}}$. We leave this analysis to further developments.

3.5 Boundary state of a non-abelian brane

In this subsection we derive the boundary state of a space-filling D25 brane of the bosonic string theory, whose worldvolume spatial dimensions are partially or totally compactified on a torus $T^{\hat{d}}$. The part of the boundary state containing the non-zero modes is the same as in the uncompactified case. Therefore we need only to determine the part with the zero modes.[3] Our results can also be easily extended to the case of the D9 brane of a superstring, because the zero mode structure is the same in the two cases.

The starting point is the computation of the annulus diagram in the open string channel and this will allow us to make contact with previous results [17, 32, 18]. In this calculation the role played by the bundle at string level, developed in the previous subsections, will be very important. Then, by using open/closed string duality, we rewrite it in the closed channel and from it we derive the boundary state. Subsequently, the same boundary will be determined directly in the closed string channel along the lines of ref. [33]. The two approaches provide the same boundary state up to a phase that, however, does not contribute to the annulus diagram.

In order to use all the machinery developed in the previous sections we restrict ourselves to the background $\mathbb{R}^{1,25-\hat{d}} \times (T^2)^{\hat{d}/2}$ ($\hat{d}$ even). This simplification, however, will not be necessary for determining the boundary state directly from the closed string channel. On each of the tori we turn on a background gauge field with constant field strength given by:

$$F_l \equiv \begin{pmatrix} 0 & F_{12}^l \\ -F_{12}^l & 0 \end{pmatrix} \mathbb{I}_{N_l}.$$

[3] In the final writing of this paper we were informed by D. Duo, R. Russo and S. Sciuto that they have obtained a similar expression for the boundary state [24].

Let us start by computing the annulus diagram in the open string channel in the case in which the open strings are attached to two space-filling branes having the same gauge field on their worldvolume and therefore satisfying the dipole condition. This amounts to evaluate:

$$\mathcal{Z}^{\text{dipole}}_{25;\,F} = M^2 \int_0^\infty \frac{d\tau}{\tau} \text{Tr}\left[e^{-2\pi\tau L_0}\right] \tag{3.24}$$

where, in order to be more general, we have considered M space-filling branes producing the factor M^2 in the previous equation. These are degrees of freedom associated to an additional $\mathrm{U}(M)$ gauge group under which the background gauge fields are uncharged. L_0 is the open string Hamiltonian given in eq. (2.26). After some calculation, eq. (3.24) in the non-degenerate case becomes:

$$\begin{aligned}\mathcal{Z}^{\text{dipole}}_{25;\,F} &= \frac{M^2 V_{26-\hat{d}}}{(8\pi^2\alpha')^{(26-\hat{d})/2}} \int_0^\infty \frac{d\tau}{\tau} \tau^{-\frac{26-\hat{d}}{2}} \left[f_1(e^{-\pi\tau})\right]^{-24} \\ &\times \prod_{l=1}^{\hat{d}/2} \left[\sum_{n^{(l)}\in\mathbb{Z}} e^{-2\pi\tau \sum_{p,q=1}^{2}\left[\frac{n_p^{(l)}}{N_l}(\mathcal{G}^{(l)})^{pq}\frac{n_q^{(l)}}{N_i}\right]}\right].\end{aligned} \tag{3.25}$$

This equation can be rewritten in the closed string channel by using the following Poisson resummation formula:

$$\sum_{w\in\mathbb{Z}^p} \exp\left[-\pi(w+x)A(w+x)\right] = (\det A)^{-1/2} \sum_{w\in\mathbb{Z}^p} \exp\left[-\pi\, wA^{-1}w + 2\pi i\, w\, x\right]$$

and the transformation properties of the function $f_1(e^{-\pi\tau}) = \sqrt{t} f_1(e^{-\pi t})$ under $\tau \to t = \frac{1}{\tau}$, obtaining:

$$\begin{aligned}\mathcal{Z}^{\text{dipole}}_{25;\,F} &= \frac{M^2 V_{26-\hat{d}}}{(8\pi^2\alpha')^{(26-\hat{d})/2}} \prod_{l=1}^{\hat{d}/2} \left[\left(\det\frac{\mathcal{G}^{(l)}_{pq}}{2}\right)^{1/2} N_l^2\right] \\ &\times \int_0^\infty dt\ \left[f_1(e^{-\pi t})\right]^{-24} \prod_{l=1}^{\hat{d}/2}\left[\sum_{s_{(l)}\in\mathbb{Z}} e^{-\frac{\pi}{2}ts^p_{(l)}N_l\mathcal{G}^{(l)}_{pq}s^q_{(l)}N_l}\right].\end{aligned} \tag{3.26}$$

Eq. (3.26) gives the interaction between two stacks of M D25 magnetized branes in the closed channel. By using the equation:

$$\mathcal{Z}^{\text{dipole}}_{25;\,F} = \langle D25(E,\,F)|\Delta|D25(E,\,F)\rangle$$

where Δ is the closed string propagator, we can determine the boundary state apart from an overall phase.

In this way we get the following expression of the boundary state, which we now write both for the D25 brane of the bosonic theory and for the D9 brane of the superstring:

$$|D(d-1)(E,\,F)\rangle = \frac{\sqrt{\det\mathcal{E}}}{(\det G)^{1/4}} M \frac{T_{d-1}}{2} e^{-\sum_{n=1}^\infty \frac{1}{n}\ \alpha^\mu_{-n}G_{\mu\nu}\tilde{\alpha}^\nu_{-n}} \tag{3.27}$$

$$\times e^{-\sum_{n=1}^{\infty}\frac{1}{n}\,\alpha_{-n}^{i}\mathcal{E}_{ik}(\mathcal{E}^{-T})^{kh}G_{hj}\tilde{\alpha}_{-n}^{j}}\,|k=0;0_a,0_{\tilde{a}}\rangle\prod_{l=1}^{\hat{d}/2}|D(d-1)(E,\,F)_{N_l}^{l}\rangle_{z.m.}.$$

Here, $d=26$ or 10, T_{d-1} is the tension of the space-filling brane given just before eq. (B.28), μ,ν are the non-compact space-time indices and

$$|D(d-1)(E,\,F)_{N_l}^{l}\rangle_{z.m.}=N_l\sum_{n^{(l)},m^{(l)}\in\mathbb{Z}}\delta_{\hat{n}_p^{(l)}-2\pi\alpha' qF_{pq}^{l}\hat{m}_q^{(l)}}|n_p^{(l)},m_q^{(l)}\rangle \tag{3.28}$$

where $\mathcal{E}$ is given in eq. (2.18). Notice that the particular structure of the delta function is the one that is also required by the overlap conditions in eq. (B.24). Finally, in order to reproduce the peculiar structure of the exponent in eq. (3.26) and also to implement that the first Chern-class is integer as dictated by eq. (3.11), we have to impose that $m_{(l)}^q=N_l s_{(l)}^q$ with $s_{(l)}^q\in\mathbb{Z}$ and we end with the following boundary state:

$$|D(d-1)(E,\,F)_{N_l}^{l}\rangle_{z.m.}=N_l\sum_{s_{(l)}\in\mathbb{Z}}|2\pi\alpha' N_l\,qF_{pq}^{l}\,s_{(l)}^q,N_l\,s_{(l)}^q\rangle \tag{3.29}$$

Before discussing the degenerate case let us compare the boundary state in eq. (3.29) with the one we have exhibited in eq. (17) of ref. [18]. They only differ from the fact that the zero modes in eqs. (3.28) and (3.29) have integer Kaluza-Klein momenta and winding numbers, while the ones in eq. (17) of ref. [18] have integer winding modes but fractional Kaluza-Klein modes. We think, however, that it is unnatural to have fractional Kaluza-Klein momenta in the closed string sector and therefore we prefer the boundary state given here which eliminates this feature.

The previous equations can be easily generalized to the degenerate case. Eq. (3.25) now becomes

$$\begin{aligned}\mathcal{Z}_{25;\,F}^{\text{dipole}}&=\frac{M^2V_{26-\hat{d}}}{(8\pi^2\alpha')^{(26-\hat{d})/2}}\left[\prod_{l=1}^{\hat{d}/2}g^{(l)}\right]\int_0^{\infty}\frac{d\tau}{\tau}\tau^{-\frac{26-\hat{d}}{2}}\left[f_1(e^{-\pi\tau})\right]^{-24}\\&\times\prod_{i=l}^{\hat{d}/2}\left[\sum_{n^{(l)}\in\mathbb{Z}}e^{-2\pi\tau\sum_{p,q=1}^{2}\left[\frac{n_p^{(l)}(\delta_1^p+g^{(l)}\,\delta_2^p)}{N_l}(\mathcal{G}^{(l)})^{pq}\frac{n_q^{(l)}(\delta_1^q+g^{(l)}\,\delta_2^q)}{N_i}\right]}\right]\end{aligned} \tag{3.30}$$

where the overall factor $\left[\prod_{l=1}^{\hat{d}/2}g^{(l)}\right]$ and the peculiar structure of the momenta are due respectively to the degeneracy and the structure of the dipole string momentum in the degenerate case.

By rewriting it in the closed string channel we get:

$$\begin{aligned}\mathcal{Z}_{25;\,F}^{\text{dipole}}&=\frac{M^2V_{26-\hat{d}}}{(8\pi^2\alpha')^{(26-\hat{d})/2}}\prod_{l=1}^{\hat{d}/2}\left[\left(\det\frac{\mathcal{G}_{pq}^{(l)}}{2}\right)^{1/2}N_l^2\right]\int_0^{\infty}dt\;\left[f_1(e^{-\pi t})\right]^{-24}\\&\times\prod_{l=1}^{\hat{d}/2}\left[\sum_{s_{(l)}\in\mathbb{Z}}e^{-\frac{\pi}{2}ts_{(l)}^{p}N_l\left(\delta_1^p+\frac{\delta_2^p}{g^{(l)}}\right)\mathcal{G}_{pq}^{(l)}\left(\delta_1^q+\frac{\delta_2^q}{g^{(l)}}\right)s_{(l)}^qN_l}\right].\end{aligned} \tag{3.31}$$

The zero-mode structure of the boundary state that is extracted from the previous equation has again the form given in eqs. (3.28) and (3.29), but now one can impose a weaker condition:

$$(2\pi\alpha')\frac{N_l}{g^{(l)}}qF^l_{12} = \frac{f^l}{g^{(l)}} \in \mathbb{Z}$$

which requires that $m^q_{(l)} = N_l/g^{(l)}\, s^q_{(l)}$. However, in order to reproduce from the boundary state the expression in eq. (3.31) for the degenerate case, we need to take $m^q_{(l)} = N_l\, s^q_{(l)}\, (\delta^1_q + \delta^2_q/g^{(l)})$. By collecting all the results we can write:

$$|D25(E,\, F)^l_{N_l}\rangle_{z.m.} = N_l \sum_{s_{(l)}\in\mathbb{Z}} |2\pi\alpha' N_l\, qF^l_{pq}\, s^q_{(l)}\, (\delta^1_q + \delta^2_q/g^{(l)}), N_l\, s_{(l)}\, (\delta^1_q + \delta^2_q/g^{(l)})\rangle \quad (3.32)$$

The generalization of the previous expression to the $D9$ brane is straightforward. It is easy to verify that the boundary state in eq. (3.32) reproduces the zero-mode contribution in eq. (3.31). Notice that the asymmetry between directions 1 and 2 is a direct consequence of the asymmetric choice for the transition function performed in eq. (3.12).

In the following we would like to explore the connection between the magnetized D25 branes carrying non-trivial gauge bundles and the T-dual systems corresponding to lower dimensional branes generically wrapping some cycles of the torus. The T-duality which we consider is the one that exchanges the Kähler structures $T^{(l)}_i$ with the complex structures $U^{(l)}_i$ of the torus T^2 defined as follows:

$$T^{(l)} \equiv T^{(l)}_1 + iT^{(l)}_2 = B^{(l)}_{12} + i\sqrt{G^{(l)}} \;\; ; \;\; U^{(l)} \equiv U^{(l)}_1 + iU^{(l)}_2 = \frac{G^{(l)}_{12}}{G^{(l)}_{11}} + i\frac{\sqrt{G^{(l)}}}{G^{(l)}_{11}}.$$

The exponential factor in eq. (3.30), which is essentially due to the zero modes of the open string Hamiltonian[4] on the torus T^2, can be written as:

$$\left(\frac{n^{(l)}_p\delta^p_1 + n^{(l)}_p g^{(l)}\delta^p_2}{N_l}\right)(\mathcal{G}^{(l)})^{pq}\left(\frac{n^{(l)}_q\delta^q_1 + n^{(l)}_q g^{(l)}\delta^q_2}{N_l}\right) = \frac{T^{(l)}_2}{U^{(l)}_2}\frac{|n^{(l)}_2 - n^{(l)}_1\frac{U^{(l)}}{g^{(l)}}|^2}{|\frac{N_l}{g^{(l)}}T - \frac{f^l}{g^{(l)}}|^2} \quad (3.33)$$

where we have used the open string metric on the torus T^2:

$$(\mathcal{G}^{(l)})^{pq} = \frac{T^{(l)}_2}{U^{(l)}_2({T^{(l)}_2}^2 + \mathcal{B}^{(l)2})}\begin{pmatrix} |U^{(l)}|^2 & -U^{(l)}_1 \\ -U^{(l)}_1 & 1\end{pmatrix} \;\; ; \;\; \mathcal{B} = B_{12} - 2\pi\alpha' qF_{12}.$$

Under the T-duality transformation, i.e. $T \leftrightarrow U$, the l.h.s. of eq. (3.33) becomes:

$$\frac{T^{(l)}_2}{U^{(l)}_2}\frac{|n^{(l)}_2 - n^{(l)}_1\frac{U^{(l)}}{g^{(l)}}|^2}{|\frac{N_l}{g^{(l)}}T - \frac{f^l}{g^{(l)}}|^2} \Longrightarrow \frac{U^{(l)}_2}{T^{(l)}_2}\frac{|n^{(l)}_2 - (w^{(l)}_1 + \frac{v^{(l)}_1}{g^{(l)}})T^{(l)}|^2}{|\frac{N_l}{g^{(l)}}U - \frac{f^l}{g^{(l)}}|^2} \quad (3.34)$$

where we have rewritten $n^{(l)}_1/g^{(l)}$ as $w^{(l)}_1 + v^{(l)}_1/g^{(l)}$ with $w^{(l)}_1 \in \mathbb{Z}$ and $v^{(l)}_1 = 0, \ldots, g^l - 1$. The T-dual zero mode Hamiltonian in the non-degenerate case can be interpreted as the one

[4]In the non-degenerate case and in the case of a squared torus this Hamiltonian appears in refs. [11 – 13]

of a lower dimensional brane wrapping respectively $(\pm N_l, \mp f^l)$ times the two one-cycles of the torus as we have discussed in the introduction. This can be seen by comparing the zero-mode Hamiltonian in the r.h.s. of eq. (3.34) for the squared torus T^2 with $B_{12} = 0$ $(T = i\frac{R_1 R_2}{\alpha'}; U = i\frac{R_2}{R_1})$, with the zero-mode Hamiltonian given for instance in section (3.1) of ref. [34].

In the degenerate case we see instead that, for $v_1^{(l)} = 0$, the open string Hamiltonian coincides with the one of a lower-dimensional brane with wrappings $n_l = \pm N_l/g^{(l)}$, $m_l = \mp f^l/g^{(l)}$ along the one-cycles of the torus. For $v_1^{(l)} \neq 0$, eq. (3.34) shows that also for zero winding $w_1^l = 0$ the open string has a minimal length and therefore the previous Hamiltonian describes the interaction between parallel branes displaced in the space transverse to their worldvolume.[5] In this case, the zero-mode contribution to the D-brane interaction, in the T-dual configuration, can be written as:

$$\mathcal{Z}_{\text{z.m.}} = \prod_{l=1}^{\hat{d}/2} \left[g^{(l)} \sum_{(k^{(l)}, w^{(l)}) \in \mathbb{Z}} \sum_{v^{(l)}=0}^{g^{(l)}-1} e^{-2\pi\tau \frac{U_2^{(l)}}{T_2^{(l)}} \frac{|k^{(l)} - (w^{(l)} + \frac{v^{(l)}}{g^{(l)}}) T^{(l)}|^2}{|n_l\, U + m_l|^2}} \right].$$

where the sum over $v^{(l)}$ suggests that the brane is not stable and decays into a stack of $g^{(l)}$ branes wrapped $m_l \equiv \mp f^l/g^{(l)}$ and $n_l \equiv \pm N_l/g^{(l)}$ times along the cycles of the tori $(T^2)^{(l)}$.[6]

In the last part of this subsection we derive the boundary state with a gauge field on it directly in the closed string channel starting from the boundary state without a gauge field and following the procedure described in ref. [33] that provides the following expression:

$$|D25(E,F)\rangle = Tr\left(P\; e^{+i\oint qA}\right)\; |D25(E, F=0)\rangle$$

where the boundary state without the gauge field is given in eqs. (B.28) and (B.29). The previous path ordering is explicitly evaluated in appendix D getting

$$\begin{aligned} |D25(E,F)\rangle = \frac{T_{25}}{2}\, N \frac{\sqrt{\det \mathcal{E}}}{(\det G)^{1/4}} \sum_s e^{-i\pi \hat{F}^<_{ij} s^i s^j} |n_i = \hat{F}_{ij} N s^j, m^i = s^i\rangle \qquad (3.35) \\ \times e^{-\sum_{n=1}^{\infty} \frac{1}{n}\, \alpha^i_{-n} \mathcal{E}_{ik} (\mathcal{E}^{-T})^{kh} G_{hj} \tilde{\alpha}^j_{-n}}\; |0_a, 0_{\tilde{a}}\rangle e^{-\sum_{n=1}^{\infty} \frac{1}{n}\, \alpha^\mu_{-n} G_{\mu\nu} \tilde{\alpha}^\nu_{-n}}\; |k=0\rangle. \end{aligned}$$

Here $\hat{F}^<_{ij} = \hat{F}^{ij}$ when $i < j$ and zero otherwise. The boundary state in eq. (3.35) differs from the one given in eq. (3.29) for a phase factor. However, this extra phase does not give any contribution to the one-loop free-energy and this is the reason why in the previous determination of the boundary it has not been possible to reveal its presence.

3.6 The string vertices

In this section we construct the vertex operators corresponding to the open strings having their endpoints on the D25 branes. We limit our analysis to the compactified part of the

[5] See for instance section (3.2) of ref. [34].

[6] See the discussion on page 84 of ref. [4].

vertex and also to the lowest state, the tachyonic one, being the generalization to higher state vertices straightforward.

In the non-degenerate case and on the simplest case of T^2, the compact part of the string vertex, describing an open-string tachyon living on a non-abelian brane is given by:

$$V(x;k) = e^{ik_i X^i(x)} \Lambda_{(k_1,k_2)}$$

where $X^i(x)$ is given in eq. (2.15) with $\sigma = 0$ and $x = e^{-i\tau}$, Λ is the Chan-Paton factor and the momentum is given by

$$(k_1, k_2) = \frac{1}{\sqrt{\alpha'}} \left(\frac{n_1}{N}, \frac{n_2}{N} \right).$$

Using eq. (B.47) it is easy to rewrite the previous equation as follows:

$$V(x;k) = e^{ik_i X^i_{L(0)}(x)} \Lambda_{(k_1,k_2)} \tag{3.36}$$

where we have neglected cocycle factors, an example of which is provided for instance by the last term in eq. (B.43). The generalization of such a vertex to the case of $(T^2)^{\hat{d}/2}$ is simply the factorized product of $\frac{\hat{d}}{2}$ operators, one for each torus T^2, with the same structure as the one given in eq. (3.36).

In sections (3.3) and (3.4) we have determined the structure of the Chan-Paton factors, respectively in the non-degenerate and degenerate case, up to a c-number factor. In particular, for the non-degenerate case, the Chan-Paton factor is given in eq. (3.14) together with eq. (3.20). Since now the Chan-Paton factor depends on the momentum, we must also remember that the vertex operator in eq. (3.36) has to satisfy the hermitian conjugation property

$$V(z;k)^\dagger = \frac{1}{z^{2h}} V(1/z; -k)$$

with h being the conformal weight, which imposes the following constraint on the Chan-Paton factor:

$$\Lambda^\dagger_{(k_1,k_2)} = \Lambda_{(-k_1,-k_2)}. \tag{3.37}$$

In order to satisfy the previous identity we must add a phase factor to the Chan-Paton factor determined above and we get:

$$\Lambda_{(k_1,k_2)} = \frac{1}{\sqrt{N}} \; e^{-i\pi N \, \alpha' \, \hat{h}_2 k_1 k_2} \left(Q_N^{N \sqrt{\alpha'} \, \hat{h}_2 k_2} P_N^{-N \sqrt{\alpha'} k_1} \right) \tag{3.38}$$

where $\hat{h}_2$ is defined in eq. (3.19). The fact that Λ in eq. (3.38) satisfies eq. (3.37) is a consequence of the relations: $P^\dagger = P^{-1}$ and $Q^\dagger = Q^{-1}$.

It is easy to check that the $\Lambda_{(k_1,\, k_2)}$ matrix satisfies the multiplication rule

$$\Lambda_{(k_1,k_2)ac} \Lambda_{(l_1,l_2)cb} = \frac{1}{\sqrt{N}} \; e^{ik\wedge l} \; \Lambda_{(k_1+l_1,k_2+k_2)ab}$$

where we have introduced the product

$$k \wedge l = -\pi\alpha' \; N\hat{h}_2(k_1 l_2 - k_2 l_1)$$

or more in general:

$$\prod_{i=1}^{M} \Lambda_{(k_1^{(i)},k_2^{(i)})} = N^{\frac{1-M}{2}} \prod_{i<j=1}^{M} e^{ik^{(i)}\wedge k^{(j)}} \Lambda_{(\sum_{i=1}^M k_1^{(i)},\, \sum_{i=1}^M k_2^{(i)})}.$$

In order to compare with an alternative description of wrapped branes that we will present in section 4, it is useful to evaluate explicitly the trace over the Chan Paton factors, that is given by:

$$\begin{aligned}\mathrm{Tr}\left[\prod_{i=1}^{M} \Lambda_{(k_1^{(i)},k_2^{(i)})}\right] &= N^{-\frac{M}{2}} \prod_{i<j=1}^{M} e^{ik^{(i)}\wedge k^{(j)}} \\ &\times \delta^{[N]}_{N\sqrt{\alpha'}\sum_{i=1}^M k_1^{(i)};0}\, \delta^{[N]}_{N\sqrt{\alpha'}\sum_{i=1}^M k_2^{(i)};0}\end{aligned} \tag{3.39}$$

The normalization coefficient $\frac{1}{\sqrt{N}}$ introduced in eq. (3.38) is there to ensure that the trace over two Chan-Paton factors is independent on the number of colors.

The analysis done so far to determine the structure of the string vertices in the non-degenerate case can be easily extended to the degenerate case. One gets again the vertex

$$V(x;k,A) = e^{ik_i X^i_{L(0)}(x)} \Lambda_{(k_1,k_2),A}$$

where the momentum is given by:

$$(k_1,k_2) = \frac{1}{\sqrt{\alpha'}}\left(\frac{n_1}{N}, \frac{n_2}{N/g}\right).$$

By analogy with eq. (3.38) we take, for the momentum dependent Chan-Paton factor, the following expression:

$$\Lambda_{(k_1,k_2),A} = \frac{1}{\sqrt{N}}\; e^{i\frac{\pi}{N}n_1\left(\hat{h}_2 n_2 + AN/g\right)} \left(Q_N^{\hat{h}_2 n_2 + A\frac{N}{g}} P_N^{-n_1}\right).$$

It satisfies the hermiticity property

$$\Lambda^\dagger_{(k_1,k_2),A} = \Lambda_{(-k_1,-k_2),-A}$$

and the multiplication rule

$$\Lambda_{(k_1,k_2),A}\Lambda_{(l_1,l_2),B} = \frac{1}{\sqrt{N}}\; e^{i\pi\frac{\sqrt{\alpha'}N}{g}(k_1 B - l_1 A)}\; e^{i\frac{1}{g}k\wedge l}\; \Lambda_{(k_1+l_1,k_2+k_2),A+B}.$$

4. Narain branes

In the previous section we have constructed the boundary state and the open string vertex operators corresponding to wrapped space-filling branes with a background gauge field living on their worldvolume. They are described by gauge bundles. However, as we have pointed out in the introduction, this is not necessarily the unique way of describing wrapped magnetized space-filling branes and in this section we discuss another kind of space-filling branes, the *Narain branes*. Their name is due to the fact that they can be obtained from the usual branes without a background gauge field by means of a transformation of the Narain T-duality group $O(\hat{d}, \hat{d}, Z)$ which is reviewed in appendix C. We construct the boundary state corresponding to this kind of branes and show that it is coincident (up to a phase which does not contribute to the one-loop vacuum amplitude) with the one already constructed in the previous section for the gauge bundles. Then we add Wilson lines to this boundary state in the case $F = 0$ in order to describe a D-brane wrapped N times around a torus and analyze their transformation properties under the Narain group, and then generalize to the case $F \neq 0$. We give the vertex operators for the open strings having their endpoints attached to the Narain branes, showing that their scattering amplitudes with closed strings are different from those that one derives from the gauge bundles. In all the examples we will explicitly refer to the tachyon vertex because it encodes the main features of the problem, the generalization to all other vertices being straightforward.

4.1 Narain branes from plain brane: general case.

In this section we consider the bosonic string, taking as our starting point a D25 brane in a generic constant closed string background E^t with no background gauge field ($F^t = 0$) on its worldvolume. By applying on it a general transformation of the T-duality group, we get what we call the most general Narain brane having a non-vanishing constant magnetic field F on its worldvolume.

We start with a plain $D25$ on $R^1 \otimes T^{25}$ whose boundary state satisfies the boundary conditions in eq. (B.23) with $F^t = 0$:

$$\left[G^t_{ij}(\dot{X}^t)^j + B^t_{ij}(X'^t)^j\right]_{\tau=0} |D25(E^t, F^t = 0)\rangle \equiv P^t_i |D25(E^t, F^t = 0)\rangle = 0 \qquad (4.1)$$

The solution of these equations is given in eqs. (B.28) and (B.29) for $d = 26$.

We now perform a canonical transformation as in eqs. (C.1), (C.2) such that $\hat{\mathcal{D}}^{-1}\hat{\mathcal{C}}$ is a well-defined quantity, then the boundary defining eq. (4.1) becomes

$$\left[P_i + (\hat{\mathcal{D}}^{-1}\hat{\mathcal{C}})_{ij}\, \frac{X'^j}{2\pi\alpha'}\right] |D25(E, F\rangle = 0 \qquad (4.2)$$

that is equal to eq. (B.23) with the following gauge field:

$$\hat{F} = 2\pi\alpha' q\; F = -\hat{\mathcal{D}}^{-1}\hat{\mathcal{C}} = \hat{\mathcal{C}}^T \hat{\mathcal{D}}^{-T}. \qquad (4.3)$$

The last equality follows from the entry $(2, 2)$ of eq. (C.13). Here q is the electric charge.

Under this transformation the zero mode part of the boundary (B.29) becomes:

$$
\begin{aligned}
|D25(E,F)\rangle_{zm} &= |D25(E^t, F^t = 0)\rangle_{zm} \\
&= \sqrt{\det \hat{\mathcal{D}}}\frac{\sqrt{\det \mathcal{E}}}{(\det G)^{1/4}} \sum_{s \in Z^{25}} |n_i = (\hat{\mathcal{C}}^T)_{ij} s^j, m^i = \hat{\mathcal{D}}^{T\,i}{}_j s^j\rangle \\
&= \sqrt{\det \hat{\mathcal{D}}}\frac{\sqrt{\det \mathcal{E}}}{(\det G)^{1/4}} \sum_{s \in Z^{25}} |n_i = 2\pi\alpha' q F_{ij} m^j, m^i = \hat{\mathcal{D}}^{T\,i}{}_j s^j\rangle
\end{aligned} \tag{4.4}
$$

where in the first equality we have written the boundary state with a non-vanishing gauge field as the one in eq. (B.29) with $n^t_i = 0$ and $m^{t\,i} = s^i$. By rewriting those variables in terms of n and m given in the upper equation in (C.15) one gets the second line of eq. (4.4) that can finally be written as in the third line by means of eq. (4.3). A detailed explanation of the normalization factor is given in ref. [18] and reviewed in appendix C. Finally, when $\det \hat{\mathcal{D}} \neq 0$, the complete boundary state (B.28) satisfying eq. (4.2) is:

$$
\begin{aligned}
|D25(E,F)\rangle = &\frac{T_{25}}{2}\sqrt{\det \hat{\mathcal{D}}}\frac{\sqrt{\det \mathcal{E}}}{(\det G)^{1/4}} \sum_{s \in Z^{25}} |n_i = 2\pi\alpha' q F_{ij} m^j, m^i = \hat{\mathcal{D}}^{T\,i}{}_j s^j\rangle \\
&\times e^{-\sum_{n=1}^{\infty}\frac{1}{n}\,\alpha^i_{-n}\mathcal{E}_{ik}(\mathcal{E}^{-T})^{kh} G_{hj}\tilde{\alpha}^j_{-n}} |0_a, 0_{\tilde{a}}\rangle e^{-\sum_{n=1}^{\infty}\frac{1}{n}\,\alpha^0_{-n} G_{00}\tilde{\alpha}^0_{-n}} |k_0 = 0\rangle.
\end{aligned} \tag{4.5}
$$

By construction this boundary state satisfies eq. (B.23) with $\hat{F}$ given in eq. (4.3). Notice also that this construction is valid for an arbitrary torus $T^{\hat{d}}$.

4.2 Special cases

T-duality on a factorized torus. In section 3 we have given the boundary state of a wrapped magnetized brane in the gauge bundle description. We would like here to compare this with the boundary state corresponding to a Narain brane. To this aim, we consider the simple case in which the compact space is a factorized torus. In particular, we can focus on a single T^2, being the generalization to $(T^2)^{\hat{d}/2}$ straightforward, and as a very special example we consider the canonical transformation acting in the first torus $T^2_{(1)}$ along the directions 1 and 2 realized by the matrix:

$$
\Lambda_2(p_{(1)},\, q_{(1)}) = \begin{pmatrix} r_{(1)}\mathbb{I} & i s_{(1)}\,\sigma_2 \\ -i p_{(1)}\,\sigma_2 & q_{(1)}\mathbb{I} \end{pmatrix}. \tag{4.6}
$$

By imposing the condition $\Lambda_2(p_{(1)}, q_{(1)}) \in O(2,2,Z)$ (see eq. (C.3)) we get:

$$
J = \begin{pmatrix} 0 & (r_{(1)}\, q_{(1)} - s_{(1)}\, p_{(1)})\mathbb{I} \\ (-s_{(1)}\, p_{(1)} + r_{(1)}\, q_{(1)})\mathbb{I} & 0 \end{pmatrix}
$$

which implies that $r_{(1)}q_{(1)} - p_{(1)}s_{(1)} = 1$. From eqs. (C.2) and (4.6) we can see that:

$$
\det \hat{\mathcal{D}} = q^2_{(1)}.
$$

This T-duality transforms a plain D-brane into a configuration of a D-brane with a gauge field strength given by (see eq. (4.3)):

$$2\pi q\alpha' F = -\hat{\mathcal{D}}^{-1}\hat{\mathcal{C}} = \begin{pmatrix} 0 & \frac{p_{(1)}}{q_{(1)}} \\ -\frac{p_{(1)}}{q_{(1)}} & 0 \end{pmatrix} = \frac{p_{(1)}}{q_{(1)}} i\sigma_2 .$$

From this equation we can see that $2\pi q\alpha' F_{12}$ is an integer number. This realizes eq. (1.3) according to the second logical possibility discussed in the Introduction, not coming the integer $q_{(1)}$ from any trace over the gauge group. The latter condition is the first hint that the Narain branes are branes wrapped $q_{(1)}$ times on the entire torus.

In this case eq. (4.2) becomes

$$[P_1 - \frac{p_{(1)}}{q_{(1)}}\frac{X'^2}{2\pi\alpha'}]|D25(E,F)\rangle = [P_2 + \frac{p_{(1)}}{q_{(1)}}\frac{X'^1}{2\pi\alpha'}]|D25(E,F)\rangle = 0$$

where the zero-mode part of the boundary is now given by (see eq. (4.4))

$$|D25(E,F)\rangle_{zm} = \frac{q_{(1)}\sqrt{\det \mathcal{E}_{(1)}}}{\left(\det G_{(1)}\right)^{1/4}} \sum_{s^1, s^2 \in Z} |n_1 = p_{(1)} s^2, n_2 = -p_{(1)} s^1, m^{1,2} = q_{(1)} s^{1,2}\rangle \tag{4.7}$$

The factor $q_{(1)}$ appearing in front of the boundary in the previous equation confirms that a Narain brane can be interpreted as a brane wrapping $q_{(1)}$ times the whole torus. Indeed the area of such an object, its Dirac-Born-Infeld action (DBI) and therefore the boundary state normalization should be proportional to $q_{(1)}$ because the brane covers $q_{(1)}$ times the compact manifold and this indeed happens in eq. (4.7).

The boundary state in eq. (4.7) coincides with the one given for the non-abelian branes, see for example eq. (3.29), with the identification $q_{(1)} = N_1$. The only difference between the two is in the phase factor written in eq. (3.35). As already stressed in the Introduction, this factor does not influence the one-loop free energy.

We can consider the more general case in which the T-duality acts on each torus $T^2_{(\alpha)}$ as in eq. (4.6) with parameters $p_{(\alpha)}, q_{(\alpha)}$, getting:

$$|D25(E,F)\rangle_{zm} = \prod_{\alpha=1}^{\frac{\hat{d}}{2}} \Bigg[q_{(\alpha)} \frac{\sqrt{\det \mathcal{E}_{(\alpha)}}}{\left(\det G_{(\alpha)}\right)^{1/4}} \sum_{s^{2\alpha},\, s^{2\alpha-1} \in Z} |n_{2\alpha-1} = p_{(\alpha)} s^{2\alpha}, n_{2\alpha} = -p_{(\alpha)} s^{2\alpha-1}\rangle$$
$$\times |m^{2\alpha-1} = q_{(\alpha)} s^{2\alpha-1},\, m^{2\alpha} = q_{(\alpha)} s^{2\alpha}\rangle \Bigg].$$

Plain Dp branes. In order to make contact with the kind of T-duality that transforms Neumann into Dirichlet boundary conditions and viceversa, we consider another particular case of the Narain T-duality group. We still consider a non-magnetized space filling brane, and on it we act with the special case of the standard T-duality, given by:

$$\Lambda = \begin{pmatrix} \mathbb{I}_p & & 0_p & \\ & 0_{d-p} & & \mathbb{I}_{d-p} \\ 0_p & & \mathbb{I}_p & \\ & \mathbb{I}_{d-p} & & 0_{d-p} \end{pmatrix} \tag{4.8}$$

with $d = 25$. This T-duality transforms a plain $D25$ into a configuration of a plain Dp

$$
\begin{aligned}
& P_i^t|D25(E^t, F^t = 0)\rangle = 0 \\
\Rightarrow \qquad & P_{i_\parallel}|Dp(E, F = 0)\rangle = X'^{i_\perp}|Dp(E, F = 0)\rangle = 0
\end{aligned}
$$

with $i_\parallel = 1, \dots p$ and $i_\perp = p+1, \dots, d$. Let us give the transformation property of the boundary state under the T-duality transformation in eq. (4.8). The non-zero modes of the boundary state transform according to eq. (C.19) with

$$
(\hat{\mathcal{C}} + \hat{\mathcal{D}}E)^T(-\hat{\mathcal{C}} + \hat{\mathcal{D}}E^T)^{-T} = \begin{pmatrix} (E_{\parallel\ \parallel})^T(E_{\parallel\ \parallel})^{-1} & 0 \\ 2E_{\perp\ \parallel}(E_{\parallel\ \parallel})^{-1} & -\mathbb{I}_{\perp\perp} \end{pmatrix}
$$

where $E_{\parallel\ \parallel} = \parallel E_{i_\parallel j_\parallel} \parallel$, $E_{\parallel\ \perp} = \parallel E_{i_\parallel j_\perp} \parallel$, and so on. The zero modes transform according to eq. (C.15). Thus the boundary state becomes:

$$
\begin{aligned}
|Dp(E, F = 0)\rangle = & \frac{T_{25}}{2}\frac{\sqrt{\det_p E_{\parallel\ \parallel}}}{(\det G)^{1/4}} \sum_{s \in Z^{25}} |n_\parallel = 0, n_\perp = s_\perp, m^\parallel = s^\parallel, m^\perp = 0\rangle \\
& \times e^{-\sum_{n=1}^\infty \frac{1}{n}\ \alpha_{-n}^{i_\parallel}(E^T_{i_\parallel k_\parallel}(E^{-1})^{k_\parallel h_\parallel} G_{h_\parallel j_\parallel}\tilde{\alpha}_{-n}^{j_\parallel}} e^{-2\sum_{n=1}^\infty \frac{1}{n}\ \alpha_{-n}^{i_\perp} E^T{}_{i_\perp k_\parallel}(E^{-1})^{k_\parallel h_\parallel}(G_{h_\parallel j_\parallel}\tilde{\alpha}_{-n}^{j_\parallel}} \\
& \times e^{+\sum_{n=1}^\infty \frac{1}{n}\ \alpha_{-n}^{i_\perp} G_{i_\perp j_\perp}\tilde{\alpha}_{-n}^{j_\perp}}\ |0_a, 0_{\tilde{a}}\rangle e^{-\sum_{n=1}^\infty \frac{1}{n}\ \alpha_{-n}^0 G_{00}\tilde{\alpha}_{-n}^0}\ |k_0 = 0\rangle
\end{aligned}
$$

where the terms with $E^T_{\perp\ \parallel}(E_{\parallel\ \parallel})^{-1}$ are present since the reflection conditions $P_\parallel|Dp(E, F = 0)\rangle = 0$ mix both $\alpha^\parallel$ and $\alpha^\perp$.

4.3 General transformation of F under T-duality

We want now to determine how F and Θ transform under T-duality. Let us start from a $D25$ brane with a constant F

$$
\left[P_i - \hat{F}_{ij}\frac{X'^j}{2\pi\alpha'}\right]|D25(E, F)\rangle = 0
$$

and then perform a T-duality transformation given by the matrix Λ^{-1} in eq. (C.11). In so doing we get:

$$
\left[(\hat{\mathcal{A}}^T - \hat{F}\hat{\mathcal{B}}^T)_{ij}P_j^t + (\hat{\mathcal{C}}^T - \hat{F}\hat{\mathcal{D}}^T)_{ij}\frac{X^{t\ \prime\ j}}{2\pi\alpha'}\right]|D25(E^t, F^t)\rangle = 0
$$

It is then easy to find that when

$$
\det(\hat{\mathcal{A}} + \hat{\mathcal{B}}\hat{F}) \neq 0
$$

the system still describes a $D25$ brane

$$
[P_i^t - \hat{F}_{ij}^t\frac{X^{t\ \prime\ j}}{2\pi\alpha'}]|D25(E^t, F^t)\rangle = 0.
$$

Indeed if $\det\left(\hat{\mathcal{A}}+\hat{F}\hat{\mathcal{B}}\right)\neq 0$, eq. (4.9) can be written as:

$$\left(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F}\right)^T\left[P^t+\left(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F}\right)^{-T}\left(\hat{\mathcal{C}}+\hat{\mathcal{D}}\hat{F}\right)^T\frac{X'^t}{2\pi\alpha'}\right]|D25(E^t,F^t)\rangle=0.$$

By comparing this equation with eq. (4.9), we get:

$$\hat{F}^t=-(\hat{\mathcal{A}}^T-\hat{F}\hat{\mathcal{B}}^T)^{-1}(\hat{\mathcal{C}}^T-\hat{F}\hat{\mathcal{D}}^T)=(\hat{\mathcal{C}}+\hat{\mathcal{D}}\hat{F})(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F})^{-1}. \tag{4.9}$$

Notice that

$$\hat{F}^t=0\Rightarrow\hat{F}=\hat{D}^{-1}\hat{C}=\hat{C}^T\hat{D}^{-T}. \tag{4.10}$$

On the other side, when $\det(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F})=0$, some directions acquire Dirichlet boundary conditions. In particular when

$$\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F}=0$$

eq. (4.9) reduces to

$$X^{t'i}|D1(E^t)\rangle=0$$

corresponding to pure Dirichlet boundary conditions.

When $\det(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F})\neq 0$ we can then evaluate how $\mathcal{E}$ transforms:

$$\mathcal{E}^t\;=\;E^{tT}+\hat{F}^t=(\hat{\mathcal{A}}+\hat{\mathcal{B}}E)^{-T}(\hat{\mathcal{C}}+\hat{\mathcal{D}}E)^T+(\hat{\mathcal{C}}+\hat{\mathcal{D}}\hat{F})(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F})^{-1}$$

where we have used eqs. (C.16) and (4.9), together with eqs. (C.13) and (C.14). One can equivalently write:

$$\mathcal{E}^t\;=\;=E^{tT}+\hat{F}^t=(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F})^{-T}\mathcal{E}(\hat{\mathcal{A}}-\hat{\mathcal{B}}E^T)^{-1}. \tag{4.11}$$

From them we deduce

$$\mathcal{G}^{t\;-1}\;=\;(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F})\mathcal{G}^{-1}(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F})^T=T(F)\;\mathcal{G}^{-1}\;T^T(F) \tag{4.12}$$

$$\Theta^t\;=\;(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F})\Theta(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F})^T+\hat{\mathcal{B}}(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F})^T=T(F)\;\Theta\;T^T(F)+\hat{\mathcal{B}}T^T(F) \tag{4.13}$$

with $\mathcal{G}$ and Θ defined in appendix A. Here we have introduced the matrix

$$T(F)=(\hat{\mathcal{A}}+\hat{\mathcal{B}}\hat{F})=(\hat{\mathcal{D}}-F^t\hat{\mathcal{B}})^{-T}=T^{t\;-1}(F^t) \tag{4.14}$$

and used the relation:

$$T(F)\hat{\mathcal{B}}^T+\hat{\mathcal{B}}T^T(F)=0.$$

Notice the important fact that Θ does **not** transform "tensorially" under a T-duality but it behaves like a connection and has a shift term.

The inverses of the previous equations can be obtained using eq. (C.11), i.e $\hat{\mathcal{A}}\leftrightarrow\hat{\mathcal{D}}^T,\hat{\mathcal{B}}\rightarrow\hat{\mathcal{B}}^T,\hat{\mathcal{C}}\rightarrow\hat{\mathcal{C}}^T$ and exchanging ^{t} quantities with those without ^{t}. In comparing this set of equations with the one we have written we find

$$(\hat{\mathcal{A}}-\hat{\mathcal{B}}E^T)=(\hat{\mathcal{D}}-E^t\hat{\mathcal{B}})^{-T},\quad(\hat{\mathcal{A}}+\hat{\mathcal{B}}E)=(\hat{\mathcal{D}}-E^{tT}\hat{\mathcal{B}})^{-T}$$

4.4 Adding Wilson lines to the boundary state

In section 3.1 we have seen that a D-brane with $F = 0$ wrapped N-times around a torus can be described as a brane with gauge group $\mathrm{U}(N)$ and a Wilson line background which induces a non trivial holonomy [8]. The boundary state description for such a brane can therefore be obtained turning on Wilson lines to the boundary state discussed in section 4.1 in the special case $F^t = 0$. More explicitly we have to compute:

$$\begin{aligned}
|D25(E^t, F^t = 0, a^t)\rangle &= \mathrm{Tr}\ P e^{iq\int_0^\pi d\sigma\ a_i^t X'^{t\ i}} |D25(E^t, F^t = 0)\rangle \\
&= \sum_{b=1}^{N} e^{i2\pi\sqrt{\alpha'} q a_i^{t\ b} \hat{m}^i} |D25(E^t, F^t = 0)\rangle \\
&= \frac{T_{25}}{2} \frac{\sqrt{\det E^t}}{(\det G^t)^{1/4}}\ |D25(E^t, F^t = 0)\rangle_{nzm} \\
&\quad \times \sum_{b=1}^{N} \sum_{s\in Z^{25}} e^{i2\pi\sqrt{\alpha'} q a_i^{t\ b} s^i} |n_i^t = 0, m^{t\ i} = s^i\rangle .
\end{aligned}$$

Let us also explore the effect of a T-duality transformation on the Wilson line itself. By performing a T-duality on the previous expression one is immediately led to:

$$\begin{aligned}
|D25(E, F, a)\rangle &= \frac{T_{25}}{2} \sqrt{\det \hat{\mathcal{D}}} \frac{\sqrt{\det \mathcal{E}}}{(\det G)^{1/4}}\ |D25(E, F)\rangle_{nzm} \\
&\quad \times \sum_{b=1}^{N} \sum_{s\in Z^{25}} e^{i2\pi\sqrt{\alpha'} q a_i^{t\ b} s^i} |n_i = \hat{\mathcal{C}}^{T\,i}{}_j s^j, m^i = \hat{\mathcal{D}}^{T\,i}{}_j s^j\rangle .
\end{aligned}$$

Since we expect the Wilson line to be multiplied by the winding $m^i = \hat{\mathcal{D}}^{T\,i}{}_j s^j$ we deduce that

$$a_i^{t\ b} \hat{\mathcal{D}}^{-T\,i}{}_j = a_j^b \Rightarrow a^{t\ b} T(F) = a^b \tag{4.15}$$

where we have used eq. (4.14) for the special case $F^t = 0$ and assumed that the transformation must be dependent on F.

In order to extend the previous discussion to the case in which both a Wilson line background and an abelian background gauge field F (proportional to the unity) are turned on, one can follow the same procedure since $A_i(x) + a_i$ can be added to the boundary either as $tr\ P e^{iq\int_0^\pi d\sigma\ (A_i + a_i) X'^{t\ i}}$ or in two steps by computing $tr\ e^{iq\int_0^\pi d\sigma\ a_i X'^{t\ i}}\ e^{iq\int_0^\pi d\sigma\ A_i X'^{t\ i}}$. By choosing the second procedure, one starts with a boundary with $F^t = 0$, then one constructs the one with $F \neq 0$ by means of a T-duality transformation with matrix Λ_0 and finally one adds Wilson lines. In this way one gets:

$$\begin{aligned}
|D25(E, F, a)\rangle &= tr\ P e^{iq\int_0^\pi d\sigma\ a_i X'^i} |D25(E, F)\rangle = \sum_{b=1}^{N} e^{i2\pi\sqrt{\alpha'} q a_i^b \hat{m}^i} |D25(E, F)\rangle \\
&= \frac{T_{25}}{2} \sqrt{\det \hat{\mathcal{D}}_0} \frac{\sqrt{\det \mathcal{E}_0}}{(\det G)^{1/4}}\ |D25(E, F)\rangle_{nzm} \\
&\quad \times \sum_{b=1}^{N} \sum_{s\in Z^{25}} e^{i2\pi\sqrt{\alpha'} q a_i^b \hat{\mathcal{D}}_0^{T\,i}{}_j s^j} |n_i = \hat{\mathcal{C}}_0^{T\,i}{}_j s^j, m^i = \hat{\mathcal{D}}_0^{T\,i}{}_j s^j\rangle
\end{aligned} \tag{4.16}$$

where $\hat{F} = \hat{\mathcal{C}}_0^T \hat{\mathcal{D}}_0^{-T}$, $|D25(E,F)\rangle_{nzm}$ is the non zero mode part of the boundary, and $\hat{m}^i$ is the winding operator.

Finally one can study the transformation properties of the Wilson lines under T-duality in the case $F \neq 0$. Performing a second T-duality transformation with matrix Λ (see (C.1)) on the previous boundary, its zero mode part becomes (see eq. (C.9))

$$
\begin{aligned}
|D25(E^t, F^t, a^t)\rangle_{zm} = & \sum_{b=1}^{N} \sum_{s \in Z^{25}} e^{i2\pi\sqrt{\alpha'} q a_i^b \hat{\mathcal{D}}_0^{T\,i}{}_j s^j} \\
& \times |n^t = (\hat{\mathcal{C}}\hat{\mathcal{D}}_0^T + \hat{\mathcal{D}}\hat{\mathcal{C}}_0^T)s, m^t = (\hat{\mathcal{A}}\hat{\mathcal{D}}_0^T + \hat{\mathcal{B}}\hat{\mathcal{C}}_0^T)s\rangle
\end{aligned}
$$

from which we get the transformation rule of the Wilson line under T-duality:

$$
a_b^T \hat{\mathcal{D}}_0^T = a_b^{tT}(\hat{\mathcal{A}}\hat{\mathcal{D}}_0^T + \hat{\mathcal{B}}\hat{\mathcal{C}}_0^T) \Rightarrow a_b^T = a_b^{tT} T(F)
$$

which is valid for the case $\det T(F) \neq 0$; when $\det T(F) = 0$ there are directions x^d with DD boundary conditions where it is not possible to add Wilson lines anymore as in (4.16) but it is possible to move the brane by $e^{i\int_0^\pi d\sigma\ \Delta_d \dot{X}^d}$.

4.5 Vertex operators and scattering amplitudes in Narain branes

In the previous subsections we have studied how the boundary state and a gauge field $\hat{F}$ living on a D25 brane transform under the most general T-duality transformation. In particular we have seen that, starting from a configuration without gauge fields on the branes, the T-duality transformation, performed by the matrix Λ given in eq. (C.2), provides a configuration with a non-zero gauge field given in eq. (4.10). Moreover also the boundary state acquires the same gauge field (eq. (4.5)). In this way one can obtain a theory with a non-zero gauge field from a theory without it. Of course, if we transform not only the operators but all quantities appearing in a string amplitude, as for instance the momenta of the external particles, nothing will change because T-duality is a symmetry of string theory.

In this section we want to extend this procedure to the vertex operators. Since we know that we are going to get a theory with a gauge field given by $\hat{F} = \hat{\mathcal{C}}^T \hat{\mathcal{D}}^{-T}$, we can immediately write the vertex operators describing the emission of open string tachyons. They are given by:

$$
\begin{aligned}
\mathcal{V}_{(0)}(x;k) &= e^{iD_0(k,\hat{F},B;\hat{p})} : e^{i\left(k_0 X_L^0(x) + k_i X_{L(0)}^i(x)\right)} : \\
\mathcal{V}_{(\pi)}(y;k) &= e^{iD_\pi(k,\hat{F},B;\hat{p})} : e^{i\left(k_0 X_L^0(x) + k_i X_{L(0)}^i(y)\right)} :
\end{aligned}
\tag{4.17}
$$

where we assume that all the spatial directions are compactified. Here $x = |x|$, $y = |y|e^{i\pi}$. We consider only one of such branes and consequently we do not need to introduce any Chan-Paton factor. The factors $e^{iD_{0,\pi}(k,\hat{F},B;\hat{p})}$ are the cocycles phase factors [36] and are necessary to have a well-defined theory of open and closed strings. They can be explicitly derived by requiring the theory to satisfy certain specific constraints such as the commutativity among open and closed string vertices and a proper behavior of the vertices under

Hermitian conjugation. Cocycles play an important role in determining whether the theory is commutative or not. However, their explicit knowledge is not crucial for discussing the main features of Narain branes and for comparing them with the non-abelian bundle description of magnetized branes. Therefore we postpone their explicit evaluation and the discussion about the commutation property of the theory to further developments.

Before going on, let us now spend a few words about how eq. (4.17) can be derived. The vertex operator for an open string tachyon is obtained by inserting in the exponent of the vertex operator the string coordinate in eq. (2.15), computed at one of the endpoints of the open string. But then, for instance in the case of $\sigma = 0$, one can use eqs. (B.47) and express it in terms of $X^i_{(0)L}$, apart from a phase that contributes to the cocycle that we are anyway not considering as explained above. A similar reasoning can also be used for the other endpoint.

Let us now consider the vertex for the closed string tachyon. If the vertex has to be used in an amplitude with only closed strings (sphere diagram), then, apart from a cocycle factor, it is given by:

$$\mathcal{W}_{T_c}(z, \bar{z}; k_L, k_R) =: e^{(ik_0 X^0(z,\bar{z}) + k_{Li} X^i_L(z) + k_{Ri} \tilde{X}^i_R(\bar{z}))} : \tag{4.18}$$

where X_L and $\tilde{X}_R$ are given respectively in eqs. (B.10) and (B.11) and the variables z and $\bar{z}$ are defined in the entire complex plane.

On the other hand, the vertex for the closed string tachyon describing interactions on the disk diagram is given by [37, 35, 36]

$$\begin{aligned} \mathcal{W}_{T_c}(z, \bar{z}; k_L, k_R, y_0) &= e^{iD_C(k,\hat{F},B;\hat{p})} \\ &: e^{i\left[\frac{1}{2}k_0 X^0_L(z) + k_{Li}(G^{-1}\mathcal{E})^i{}_j X^j_{L(0)}(z)\right]} :: e^{i\left[\frac{1}{2}k_0 X^0_R(\bar{z}) + k_{Ri}(G^{-1}\mathcal{E}^T)^i{}_j X^i_{R(0)}(\bar{z})\right]} : \end{aligned} \tag{4.19}$$

Here z is defined in the upper half complex plane and the phase factor $e^{iD_C(k,\hat{F},B;\hat{p})}$ are the closed string cocycles. After having determined the open and closed string vertex operators, one can then compute the scattering amplitudes involving them, but, before doing that, let us first describe the action of T-duality on both the closed and open string vertex operators. This will allow us to rederive eqs. (4.17) and, more importantly, to study in which sense the Narain branes are wrapped branes.

Let us start from the closed string vertex given in eq. (4.18) and show that, under a T-duality transformation, it keeps the same form. By considering of course only the compact space part and extending the T-duality transformations in eqs. (C.6) to be valid also for the coordinates X^i_L and $\tilde{X}^i_R$, and not just for their derivatives, we get:

$$X^{ti}_L = (\mathcal{A} + \mathcal{B}E) X^i_L \quad ; \quad X^{ti}_L = (\mathcal{A} - \mathcal{B}E^T)\tilde{X}^i_R.$$

On the other hand, we have to remember that also the external momenta k_L and k_R transform according to eqs. (C.7) that, with eqs. $k_{L,Ri} = G_{ij} k^j_{L,R}$ and (C.17), imply the following transformations:

$$k^T_{Li} = k^{t\,T}_{Lj} (\hat{\mathcal{A}} + \hat{\mathcal{B}}E)^j{}_i \qquad k^T_{Ri} = k^{t\,T}_{Rj} (\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)^j{}_i. \tag{4.20}$$

By using the two previous equations, it is easy to see that the exponent in the vertex operator remains the same in form:

$$k^{t\ T}_{Li} X^{ti}_L + k^{t\ T}_{Ri} \tilde{X}^{ti}_R = k^T_{Li} X^i_L + k^T_{Ri} \tilde{X}^i_R$$

where the index T has been introduced here for the sake of clarity.

Let us consider now the vertex for the closed string tachyon to be used on a disk amplitude given in eq. (4.19). From the transformation properties of the closed string momenta given in eqs. (4.20) one can deduce the transformations under T-duality of the left and right moving parts of the vertex operator by requiring that the closed string vertices, written in open string formalism, are invariant in form under such a transformation. In this way we get:

$$\begin{aligned} G^{t\ -1}\mathcal{E}^t X^t_{L(0)}(z) &= (\hat{\mathcal{A}} + \hat{\mathcal{B}}E)G^{-1}\mathcal{E} X_{L(0)}(z), \\ G^{t\ -1}\mathcal{E}^{t\ T} X^t_{R(0)}(\bar{z}) &= (\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)G^{-1}\mathcal{E}^T X_{R(0)}(\bar{z}) \end{aligned} \tag{4.21}$$

for the left and right components of X.

But we have to take into account that in the open string formalism the left and right parts are not independent because of the reflection conditions. They can be of two types either Neumann or Dirichlet.

One can therefore distinguish two cases:

1. The reflection conditions are generalized Neumann boundary conditions[7] in both the original theory and in the T-dual one. This implies the two equations:

$$X^t_{L(0)}(x) = X^t_{R(0)}(x) \quad ; \quad X_{L(0)}(x) = X_{R(0)}(x). \tag{4.22}$$

Thus, after imposing $X^t_{L(0)}(x) = X^t_{R(0)}(x)$ and using $X_{L(0)}(x) = X_{R(0)}(x)$ in eq. (4.21), one gets:

$$\begin{aligned} \left(G^{t\ -1}\mathcal{E}^t\right)^{-1}(\hat{\mathcal{A}} + \hat{\mathcal{B}}E)G^{-1}\mathcal{E} &= \left(G^{t\ -1}\mathcal{E}^{t\ T}\right)^{-1}(\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)G^{-1}\mathcal{E}^T \\ &= \hat{\mathcal{A}} + \hat{\mathcal{B}}F = T(F). \end{aligned}$$

The third identity can be verified by using eqs. (C.17) and (4.11). From eqs. (4.21) we find:

$$X^t_{L(0)}(z) = T(F)\ X_{L(0)}(z) \qquad X^t_{R(0)}(\bar{z}) = T(F)\ X_{R(0)}(\bar{z}) \tag{4.23}$$

which can be simply interpreted as due to the fact that in the T-dual system distances are rescaled by $T(F)$. Thus the boundary conditions in eq. (4.22) can both be imposed when

$$\det T(F) = \det(\hat{\mathcal{A}} + \hat{\mathcal{B}}\hat{F}) \neq 0.$$

After having examined the closed string vertices, we can now discuss what happens to the open string vertices in eqs. (4.17). If we require them to remain invariant in form, the following equation has to be imposed:

$$e^{ik^{t\ T} X^t_{L(0)}(x)} = e^{ik^{t\ T} T(F)\ X_{L(0)}(x)} = e^{ik^T X_{L(0)}(x)}$$

[7] We call these b.c. generalized Neumann because they are Neumann b.c. on the $X_{(0)}$ fields but not on X ones; this in the spirit of the asymmetric rotation of [12].

where we have used eq. (4.23) and

$$k^T = k^{t\ T} T(F). \tag{4.24}$$

This is the vertex that we have already written down in eq. (4.17). The transformations in eq. (4.23) are obviously consistent with the OPEs in eq. (B.48) when

$$\mathcal{G}^t = T^{-T}(F)\mathcal{G}T^{-1}(F)$$

which matches perfectly the first equation in (4.12).

2. The reflection conditions are generalized Neumann boundary conditions in the original theory and mixed generalized Neumann and Dirichlet boundary conditions in the T-dual one. This is the case when

$$\det T(F) = \det(\hat{\mathcal{A}} + \hat{\mathcal{B}}\hat{F}) = 0.$$

This condition is very general and corresponds to various different generalized Neumann and Dirichlet boundary conditions according to the number of zero eigenvalues of the matrix T. For our discussion we consider a special subcase characterized by the following condition:

$$T = \hat{\mathcal{A}} + \hat{\mathcal{B}}\hat{F} = 0 \Rightarrow \hat{F}^t = \infty.$$

This special case corresponds to the Dirichlet reflection conditions in all the compact X^t coordinates:

$$X^t_{L(0)}(x) = -X^t_{R(0)}(x) \quad ; \quad X_{L(0)}(x) = X_{R(0)}(x). \tag{4.25}$$

However in this case the T-dual coordinate does not have anymore the expansion in eqs. (B.45) and (B.46), but one simply has (up to cocycles)

$$\hat{X}^t_L(z) = X^t_{L(0)}(z) \quad ; \quad \hat{X}^t_R(\bar{z}) = X^t_{R(0)}(\bar{z}).$$

Comparing the vertices in the two T-dual theories as in eq. (4.24) yields:

$$\begin{aligned} X^t_{L(0)}(x) &= (\hat{\mathcal{A}} + \hat{\mathcal{B}}E)G^{-1}\mathcal{E}X_{L(0)}(x); \\ X^t_{R(0)}(x) &= (\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)G^{-1}\mathcal{E}^T X_{R(0)}(x). \end{aligned}$$

Hence, the boundary conditions in eq. (4.25) are consistent if the following equation holds:

$$(\hat{\mathcal{A}} + \hat{\mathcal{B}}E)G^{-1}\mathcal{E} = -(\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)G^{-1}\mathcal{E}^T = \hat{\mathcal{B}}\mathcal{G}.$$

This can be verified with the help of $\hat{F} = -\hat{\mathcal{B}}^{-1}\hat{\mathcal{A}}$. Therefore we get a relation between the "momentum" (actually distance) in Dirichlet directions k^t and the momentum k, given by:

$$k^T = k^{tT}\hat{\mathcal{B}}\mathcal{G}.$$

Moreover, in both cases, by using eqs. (C.6) and (C.17), we have

$$T(z) = -\frac{1}{\alpha'}\partial X_L^T(z) G \partial X_L(z) = -\frac{1}{\alpha'}\partial X_L^{tT}(z) G^t \partial X_L^t(z)$$

and, because of this, all conformal dimensions are preserved.

We are now ready to compute the scattering amplitudes involving open and closed strings in the case of non-abelian and Narain branes, and compare them. As we will see below, the amplitudes exhibiting a difference between the non-abelian and Narain branes are the ones involving both open and closed strings. In particular, it is sufficient to limit ourselves to external tachyons. We only consider the compact part of correlators involving one closed string tachyon and M open string tachyons, up to phases that we have systematically neglected in this paper. In the case of non abelian branes one gets

$$\begin{aligned}&\langle 0|W(z,\bar{z};k_L,k_R)V(x_1;k_1)\ldots V(x_M;k_M)|0\rangle_{\text{compact}}\\ &\qquad = \mathrm{Tr}\left[\prod_{i=1}^{M}\Lambda_{(k_1^{(i)},k_2^{(i)})}\right] A(z,\bar{z},x_r,k_i)\,\delta_{\mathcal{E}^T G^{-1}k_L+\mathcal{E}G^{-1}k_R+\sum_{r=1}^{M}k_r,0}\end{aligned} \tag{4.26}$$

while for the Narain branes one gets

$$\begin{aligned}&\langle 0|W(z,\bar{z};k_L,k_R)V(x_1;k_1)\ldots V(x_M;k_M)|0\rangle_{\text{compact}}\\ &\qquad = A(z,\bar{z},x_r,k_i)\,\delta_{\mathcal{E}^T G^{-1}k_L+\mathcal{E}G^{-1}k_R+\sum_{r=1}^{M}k_r,0}\end{aligned}$$

where

$$\begin{aligned}A(z,\bar{z},x_r,k_i) \equiv &\prod_{r=1}^{M}(z-x_r)^{2\alpha' k_L^T G^{-1}\mathcal{E}\mathcal{G}^{-1}k_r}(\bar{z}-x_r)^{2\alpha' k_R^T G^{-1}\mathcal{E}^T\mathcal{G}^{-1}k_r}\\ &\times\prod_{1=r<s}^{M}(x_r-x_s)^{2\alpha' k_r^T\mathcal{G}^{-1}k_s}\,(z-\bar{z})^{2\alpha' k_L^T\mathcal{E}^{-T}\mathcal{E}G^{-1}k_R}\,.\end{aligned} \tag{4.27}$$

Eqs. (4.26) and (4.27) are easily seen to differ only for the trace over the Chan Paton factors which is given in eq. (3.39). This is the consequence of the fact that the vertex operators for closed string tachyons is the one given in eq. (4.18) for both theories, while the ones for open string tachyons have the same operatorial part as those in eq. (4.17), but those in the non-abelian branes have in addition momentum dependent Chan-Paton factors, given in eq. (3.38).

By using the formulas in the appendices it is easy to show that the δ-function which is common to the two correlators gives:

$$n_i - \hat{F}_{ij}m^j + \sum_{r=1}^{M}\frac{n_i^r}{N} = 0 \;\;;\;\; i=1,2$$

in terms of the momentum n_i and winding number m^i of the closed string and the momenta $\frac{n_i^r}{N}$ of the open strings. For the sake of simplicity we have restricted our analysis to the two direction of a torus T^2. Using eq. (1.2) in the previous equation one gets:

$$n_1 - \frac{mm^2}{N} + \sum_{r=1}^{M}\frac{n_1^r}{N} = n_2 + \frac{mm^1}{N} + \sum_{r=1}^{M}\frac{n_2^r}{N} = 0 \tag{4.28}$$

They can be satisfied only if the following relations holds

$$\sum_{r=1}^{M} n_1^r - mm^2 = s_1 N \quad ; \quad \sum_{r=1}^{M} n_2^r + mm^1 = s_2 N \tag{4.29}$$

where s_1 and s_2 are arbitrary integers. Finally inserting them back in eq. (4.28) one gets:

$$n_1 + s_1 = n_2 + s_2 = 0 \tag{4.30}$$

The constraint imposed by the δ-function can be satisfied only if eqs. (4.29) and (4.30) are satisfied

In the case of non-abelian branes the trace over the Chan Paton factors imposes the following additional constraints:

$$\sqrt{\alpha'} N \sum_{r=1}^{M} k_{1,2}^r = \sum_{r=1}^{M} n_{1,2}^r = r_{1,2} N$$

where $r_{1,2}$ are integer numbers.

In conclusion, for non-abelian branes the relations in eq. (4.31) must be considered together with eqs. (4.29) and (4.30). Therefore the class of solutions that one gets for the Narain branes is bigger than the one for the non-abelian branes and this means that the two theories are not equivalent. Notice, however, that a difference can be noticed only if the scattering amplitude involves at least one closed string. If we had only open strings then one would get precisely the same conditions for the two cases. The same is true for the case of a closed string with $n_i = m^i = 0$.

In the previous section we have shown that the non-abelian branes provide a description of branes wrapped N times on the torus T^2 through the introduction of a non-abelian gauge bundle based on the gauge group $\mathrm{U}(N)$. In other words, the wrapping number N is provided by the order of the gauge group. This is the reason why for this kind of branes we must introduce Chan-Paton factors that turned out to be momentum dependent. In the case of the Narain brane we do not have any non-abelian gauge field. Then in what sense do the Narain branes provide a description of wrapped branes? Or can we say that the Narain branes provide an alternative description of them? And if yes, what is the precise meaning to give to this claim?

In the introduction we have discussed two possibilities for obtaining eq. (1.2). The first one is based on the presence of a non-abelian gauge bundle and this is the one realized by the non-abelian branes. In the following, we aim to show that the Narain branes seem to realize the other possibility discussed around eq. (1.3). In order to see how this comes about, we have to study what happens to the open string coordinate when we go around the torus. This is what we are going to discuss in the last part of this subsection.

We start rewriting the tachyon open string vertex operator in eq. (4.17) for a Narain brane which is obtained through a T-duality from a plain brane with $F^t = 0$, as follows:

$$V_{(0)T,c}(x;k) \sim : e^{ik_i^t (\hat{\mathcal{D}}^{-T})^i{}_j X^j_{L(0)}(x)} :$$

where we used $T(F) = \hat{\mathcal{D}}^{-T}$ (because of eq. (4.14)). The fact that $\sqrt{\alpha'}k_i^t \in Z$ immediately implies that the theory is invariant under

$$X \to X + 2\pi\sqrt{\alpha'}\hat{\mathcal{D}}^T s \quad \forall s \in Z^{25}$$

while in the original theory the vertex operator was only invariant under

$$X^t \to X^t + 2\pi\sqrt{\alpha'}s.$$

The periodicity of the open string coordinate can also be verified directly starting from the operator which performs a shift of $X^t \to X^t + 2\pi s$ and rewriting it in the T-dual theory

$$\mathcal{T}_s^t = e^{2\pi i \ s^T\mathcal{G}^t p^t} = e^{2\pi i \ s^T\mathcal{G}^t T(F)p} = e^{2\pi i \ s^T T^{-T}(F)\mathcal{G}p} = e^{2\pi i \ (\hat{\mathcal{D}}^T s)^T\mathcal{G}p} = \mathcal{T}_{\hat{\mathcal{D}}^T s}.$$

In order to see more explicitly what happens it is convenient to specialize the previous discussion to the first case treated in section 4.2. There, T-duality acts on each torus $T^2_{(t)}$ as in eq. (4.6) with parameters $(p_{(t)}, q_{(t)})$. It is straightforward to write the compact part of the tachyonic vertex, in fact, by focusing only on the first torus:

$$V_{(0)T,c}(x;k) \sim: e^{i\frac{n_1 X^1_{L(0)}(x) + n_2 X^2_{L(0)}(x)}{\sqrt{\alpha'}\ q_{(1)}}}:$$

where the compact momentum is $\sqrt{\alpha'}\ k = \left(\frac{n_1}{q_{(1)}}, \frac{n_2}{q_{(1)}}\right)$ with all n integers.

Unlike the open string vertices in the case of non-abelian branes, these vertices have no Chan-Paton factors and describe objects with non-trivial wrapping

$$X^{1,2} = X^{1,2} + 2\pi\sqrt{\alpha'}q_{(1)}s.$$

As discussed in section 4.2, the normalization factor in front of the boundary state in eq. (4.7) suggests that a Narain brane is a brane wrapped q times around the whole torus. This means that the previous periodicity conditions have to be interpreted as simultaneous conditions on X^1 and X^2 (in the case of T^2)

$$(X^1, X^2) = (X^1 + 2\pi\sqrt{\alpha'}q_{(1)}s, X^2 + 2\pi\sqrt{\alpha'}q_{(1)}s)$$

while

$$(X^1, X^2) \neq (X^1, X^2 + 2\pi\sqrt{\alpha'}q_{(1)}s) \qquad (X^1, X^2) \neq (X^1 + 2\pi\sqrt{\alpha'}q_{(1)}s, X^2)$$

This is consistent with the fact that, with the special choice of the T-duality transformation given in eq. (4.6), the matrix $\hat{\mathcal{D}}$ is purely diagonal with two identical entries.

In conclusion, the theory based on the Narain branes seems to provide a description of branes wrapped on the two-cycle of the torus, that is different, rather than alternative, from that provided by the non-abelian branes. A further study of these two different formulations of wrapped branes is needed to better clarify their physical properties and what kind of wrapped branes they really describe.

A. Conventions

- Indices:
  Non-compact $\mu, \nu = 0, \dots 25 - \hat{d}$;

  Compact $i, j, \dots = 1, \dots \hat{d}$;

- $\delta^{[N]}_{m,n}$ means $m \equiv n \quad mod\ N$;

- 't Hooft matrices P_N and Q_N:

$$P_N = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \\ 0 & & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix} \quad ; \quad Q_N = e^{\frac{\pi i(1-N)}{N}} \begin{pmatrix} 1 & \dots & & 0 \\ 0 & e^{\frac{2i\pi}{N}} & \dots & \\ \vdots & \ddots & & \vdots \\ 0 & \dots & & e^{2i\pi\frac{(N-1)}{N}} \end{pmatrix}$$

  satisfying the commutation relation:

$$P_N \, Q_N = Q_N \, P_N \, e^{2\pi \, i/N}. \tag{A.1}$$

- Background matrices:

$$\begin{aligned} E &= \parallel E_{ij} \parallel = G + B \\ \mathcal{E} &= \parallel \mathcal{E}_{ij} \parallel = E^T + 2\pi\alpha' q_0 F = G - \mathcal{B} \end{aligned} \tag{A.2}$$

  and

$$\begin{aligned} \hat{F} &= 2\pi\alpha' q_0 F \\ \mathcal{B} &= B - 2\pi\alpha' q_0 F = B - \hat{F} \\ \mathcal{E}^{-1} &= \mathcal{G}^{-1} - \Theta \end{aligned}$$

  from which we deduce that

$$\begin{aligned} \mathcal{E}\mathcal{G}^{-1}\mathcal{E}^T &= \mathcal{E}^T\mathcal{G}^{-1}\mathcal{E} = G \\ \Theta &= \frac{1}{2}\left(\mathcal{E}^{-T} - \mathcal{E}^{-1}\right) = -\mathcal{E}^{-1}\mathcal{B}\mathcal{E}^{-T} \end{aligned}$$

B. Review of open and closed strings in flux background

In this appendix we review the solution of closed string equations of motion in constant backgrounds on a torus in order to fix our notations and give some technical details about the open string solution as well.

B.1 Action and equations of motion

Let us consider the action for the spatial coordinates, labelled by the indices $a, b = 1, \ldots, d-1$, of a bosonic string[8] interacting with a constant gravitational and a Kalb-Ramond background that is given by eq. (2.2).

Constant gravitational and Kalb-Ramond fields naturally arise when considering string theory on a $\hat{d}$-dimensional torus $T^{\hat{d}}$.[9] Toroidal compactification requires the following equivalence relation to be satisfied by any point x^i $(i = 1, \ldots, \hat{d})$ of the torus $T^{\hat{d}}$:

$$x^i \equiv x^i + 2\pi\sqrt{\alpha'}m^i \tag{B.1}$$

where m^i is an arbitrary integer. This relation has to be satisfied also by the string coordinates themselves:

$$X^i \equiv X^i + 2\pi\sqrt{\alpha'}m^i. \tag{B.2}$$

The classical equation of motion for the string coordinates derived from S is given by the usual free two-dimensional wave-equation:

$$\partial_\alpha \partial^\alpha X^j = 0. \tag{B.3}$$

In order that the action be stationary under the general variation $X^i \to X^i + \delta X^i$ we must also impose either the closed string boundary condition

$$X^i(\tau, \sigma + \pi) \equiv X^i(\tau, \sigma) \tag{B.4}$$

or one of the two boundary conditions at $\sigma = 0$:

$$\begin{gathered} X^i|_{\sigma=0} = const \\ G_{ij}\partial_\sigma X^j + B_{ij}\partial_\tau X^j|_{\sigma=0} = 0. \end{gathered} \tag{B.5}$$

and similarly, and independently, at $\sigma = \pi$.

In the presence of such non trivial backgrounds the string conjugate momentum density turns out to be:

$$P_i \equiv \frac{\partial L}{\partial \dot{X}^i} = \frac{1}{2\pi\alpha'}\left[G_{ij}\dot{X}^j + B_{ij}X'^j\right] \quad \Rightarrow \quad \dot{X}^i = 2\pi\alpha' G^{ij}P_j - G^{ik}B_{kj}(X')^j \tag{B.6}$$

and the Hamiltonian is given by:

$$H = \int_0^\pi d\sigma \left[P_i(\dot{X})^i - L\right] = \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma G_{ij}\left(\dot{X}^i\dot{X}^j + X'^i X'^j\right). \tag{B.7}$$

By plugging eq. (B.6) in eq. (B.7) we get:

$$H = \pi \int_0^\pi d\sigma \left[\alpha' P_i G^{ij} P_j + \frac{1}{\pi}(X')^i B_{ij} G^{jk} P_k + \frac{1}{(2\pi)^2\alpha'}(X')^i \left(G_{ij} - B_{ik}G^{kh}B_{hj}\right)(X')^j\right]. \tag{B.8}$$

[8] Although we consider the bosonic string where $d = 26$, we leave d arbitrary because many of the results are also valid for the superstring where $d = 10$.

[9] We assume that $\hat{d}$ spatial coordinates are compact, while the remaining $d - 1 - \hat{d}$ are non-compact.

B.2 General solution for the closed string

The general solution of (B.3) compatible with the closed-string boundary condition (B.4) is:[10]

$$X^i(\sigma,\tau) = \frac{1}{2}\left(X^i_L(\tau+\sigma) + \tilde{X}^i_R(\tau-\sigma)\right) \tag{B.9}$$

where the left and right moving parts are defined as follows:

$$X^i_L(\tau+\sigma) = x^i_L + 4\alpha' G^{ij} p_{Lj}(\tau+\sigma) + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha^i_n e^{-2in(\tau+\sigma)}, \tag{B.10}$$

$$\tilde{X}^i_R(\tau-\sigma) = x^i_R + 4\alpha' G^{ij} p_{Rj}(\tau-\sigma) + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^i_n e^{-2in(\tau-\sigma)}. \tag{B.11}$$

One has:

$$x^i = x^i_L = x^i_R \quad ; \quad \frac{p_i}{2} = p_{Li} = p_{Ri}$$

in non-compact directions and

$$\begin{aligned} p_{Li} &= \frac{1}{2\sqrt{\alpha'}}\left[\sqrt{\alpha'}p_i - B_{ij}m^j + G_{ij}m^j\right] \\ p_{Ri} &= \frac{1}{2\sqrt{\alpha'}}\left[\sqrt{\alpha'}p_i - B_{ij}m^j - G_{ij}m^j\right] \end{aligned} \tag{B.12}$$

in compact directions, where $m^i \in Z$ is the winding number. We can invert those relations getting

$$m^i = \sqrt{\alpha'}G^{ij}(p_{Li} - p_{Ri}) \qquad p_i = E_{ij}G^{jk}p_{Lk} + (E^T)_{ij}G^{ik}p_{Rk}$$

where we have defined:

$$E_{ij} \equiv G_{ij} + B_{ij}.$$

By expressing the conjugate momentum in eq. (B.6) in terms of the oscillators one gets:

$$\begin{aligned} P_i &= \frac{p_i}{\pi} + \frac{1}{\pi\sqrt{2\alpha'}}\sum_{n\neq 0}\left[(G_{ij} - B_{ij})\tilde{\alpha}^j_n e^{-2in(\tau-\sigma)} + (G_{ij} + B_{ij})\alpha^j_n e^{-2in(\tau+\sigma)}\right] \\ &= \frac{1}{2\pi\alpha'}\left[E_{ij}\partial_+ X^i + (E^T)_{ij}\partial_- X^i\right] \end{aligned} \tag{B.13}$$

where also the following relation has been used:

$$\begin{aligned} \frac{\partial X^i}{\partial\sigma} &= 2m^i\sqrt{\alpha'} + \sqrt{2\alpha'}\sum_{n\neq 0}\left[-\tilde{\alpha}^i_n e^{-2in(\tau-\sigma)} + \alpha^i_n e^{-2in(\tau+\sigma)}\right] \\ &= \partial_+ X^i - \partial_- X^i. \end{aligned} \tag{B.14}$$

Of course, along the non-compact directions one has to set $m^i \equiv 0$.

[10] With respect to the notation used in [18] we have exchd $\alpha \leftrightarrow \tilde{\alpha}$.

The quantization of the theory is obtained by imposing the following commutation relations:

$$[x^i_L, p_{Lj}] = iG^i{}_j \qquad [x^i_R, p_{Rj}] = iG^i{}_j$$
$$[\alpha^i_m, \alpha^j_n] = mG^{ij}\delta_{n+m,0} \qquad [\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] = mG^{ij}\delta_{n+m,0}$$

Those for the non-zero modes follow from imposing the canonical commutation relations, while those involving the zero modes are a consequence of the canonical commutation relations and of T-duality that requires to consider operators both $x_{L,R}$ and $p_{L,R}$ and not only the combinations $x_R + x_L$ and $p_R + p_L$.

In a compact space, like $T^{\hat{d}}$, the total momentum $p_i = \int_0^\pi d\sigma\ P_i$ has to be quantized since all physical states must be translational invariant under the shift in eq. (B.1), hence for all compact directions i one has:

$$\sqrt{\alpha'} p_i = n_i \in Z. \tag{B.15}$$

By inserting the expansions in terms of the oscillators in the Hamiltonian (B.8) one gets that the spectrum is given by the following quantity:[11]

$$\frac{H}{2} = \frac{1}{2} Z^T M Z + \frac{1}{2} \sum_{n>0} G_{ij} : \left[\alpha^i_{-n}\alpha^j_n + \alpha^i_n\alpha^j_{-n} + \tilde{\alpha}^i_{-n}\tilde{\alpha}^j_n + \tilde{\alpha}^i_n\tilde{\alpha}^j_{-n}\right] : \tag{B.16}$$

where

$$Z \equiv \begin{pmatrix} \hat{n}_j \\ \hat{m}^j \end{pmatrix} \quad , \quad M \equiv \begin{pmatrix} G^{ij} & -G^{ik}B_{kj} \\ B_{ik}G^{kj} & G_{ij} - B_{ik}G^{kh}B_{hj} \end{pmatrix}. \tag{B.17}$$

being $\hat{n}_i$ and $\hat{m}^i$ operators. It is also easy to see that:

$$\frac{1}{2} Z^T M Z = \frac{1}{2}\left[\hat{n}_i G^{ij}\hat{n}_j - 2\hat{n}_i G^{ij} B_{jk}\hat{m}^k + \hat{m}^i\left(G_{ij} - B_{ik}G^{kh}B_{hj}\right)\hat{m}^j\right]. \tag{B.18}$$

The Hamiltonian can also be written as follows:

$$\frac{H}{2} = L_0 + \tilde{L}_0 \tag{B.19}$$

with the explicit expressions of L_0 and $\tilde{L}_0$ given by:

$$L_0 = \alpha' p_{Li} p_{Lj} G^{ij} + \sum_{n=1}^{\infty} G_{ij} : \alpha^i_{-n}\alpha^j_n : \quad ; \quad \tilde{L}_0 = \alpha' p_{Ri} p_{Rj} G^{ij} + \sum_{n=1}^{\infty} G_{ij} : \tilde{\alpha}^i_{-n}\tilde{\alpha}^j_n : \tag{B.20}$$

where

$$p_{\begin{pmatrix} Li \\ Ri \end{pmatrix}} = \frac{1}{2\sqrt{\alpha'}}\left[\hat{n}_i - B_{ij}\hat{m}^j \pm G_{ij}\hat{m}^j\right]. \tag{B.21}$$

[11] From now on we consider the quantity $\frac{H}{2}$ instead of just H because it is this quantity that determines the spectrum of the theory with the correct normalization.

JHEP11(2007)100

It is straightforward to check that the level matching condition is given by;

$$\tilde{L}_0 - L_0 = \hat{n}_i \hat{m}^i + \sum_{n=1}^{\infty} G_{ij} \left[\alpha^i_{-n}\alpha^j_n - \tilde{\alpha}^i_{-n}\tilde{\alpha}^j_n\right] = 0. \tag{B.22}$$

The vacuum is then defined as the state satisfying the following conditions

$$p^i_L|0,\tilde{0}\rangle = p^i_R|0,\tilde{0}\rangle = \alpha^i_n|0,\tilde{0}\rangle = \tilde{\alpha}^i_n|0,\tilde{0}\rangle = 0 \qquad \forall n > 0$$

The momentum states are normalized as

$$\langle n_i, m^i | n'_i, m'^i\rangle = 2\pi\sqrt{\alpha'}\delta_{n_i,n'_i}\delta_{m^i,m'^i}$$

for any compact direction x^i and

$$\langle k_\mu | k'_\mu\rangle = 2\pi\delta(k_\mu - k'_\mu)$$

for any non-compact spatial direction x^a ($a = \mu \neq i$) and for the time direction ($\mu = 0$).

In the following we will consider the boundary state corresponding to a space filling brane. In this case, if one starts from eq. (2.14) with the substitution $\sigma \leftrightarrow \tau$, one can write the equation that the boundary state has to satisfy, namely:

$$\left[G_{ij}\partial_\tau X^j + (B_{ij} - 2\pi\alpha' qF_{ij})\partial_\sigma X^j\right]_{\tau=0}|B\rangle = 0. \tag{B.23}$$

In eq. (B.23) one has to insert the general solution of the classical equations of motion (B.3) compatible with the closed string boundary condition $X^i(\tau, \sigma + \pi) \equiv X^i(\tau, \sigma)$. Such solution is given in eq. (B.9).

In doing that one gets the following conditions

$$(\hat{n}_i - 2\pi\alpha' qF_{ij}\hat{m}^j)|B\rangle = 0 \tag{B.24}$$

and

$$\left[\left(G_{ij} - B_{ij} + 2\pi\alpha' qF_{ij}\right)\tilde{\alpha}^j_n + \left(G_{ij} + B_{ij} - 2\pi\alpha' qF_{ij}\right)\alpha^j_{-n}\right]|B\rangle = 0 \tag{B.25}$$

that can also be written as follows (by using eq. (B.32)):

$$\left(\mathcal{E}_{ij}\tilde{\alpha}^j_n + \mathcal{E}^T_{ij}\alpha^j_{-n}\right)|B\rangle = 0, \tag{B.26}$$

being $qF = q_0F_0$ on the boundary at $\sigma = 0$ and $qF = q_\pi F_\pi$ on the boundary at $\sigma = \pi$.

It is easy to rewrite eq. (B.24) as follows:

$$\left[\mathcal{E}_{ij}p^j_R + \mathcal{E}^T_{ij}p^j_L\right]|B\rangle = 0 \tag{B.27}$$

where $\mathcal{E}_{ij}$ is defined in eq. (2.18) and $p^i_{L,R} = G^{ij}p_{j;L,R}$.

For later use here we give the explicit form of the boundary state for a D25 brane that satisfies eqs. (B.26) and (B.27) with $F_{ij} = 0$ ($T_p = \frac{\sqrt{\pi}}{2^{\frac{d-10}{4}}}(2\pi\sqrt{\alpha'})^{\frac{d}{2}-2-p}$) :

$$\begin{aligned} |D25\rangle \; = \; & \frac{T_{25}}{2} e^{-\sum_{n=1}^{\infty}\frac{1}{n}\ \alpha^{t\ i}_{-n}(E^T)_{ik}(E^{-1})^{kh}G_{hj}\tilde{\alpha}^{t\ j}_{-n}}\ |D25\rangle_{zm,c} \\ & \times e^{-\sum_{n=1}^{\infty}\frac{1}{n}\ \alpha^{t\ 0}_{-n}G_{00}\tilde{\alpha}^{t\ 0}_{-n}}\ |k_0 = 0\rangle \end{aligned} \tag{B.28}$$

where the time direction has been added and the compact zero modes part is given by:

$$|D25\rangle_{zm,c} = \frac{\sqrt{\det E}}{(\det G)^{1/4}} \sum_{s\in Z^{25}} |n_i = 0, m^i = s^i\rangle. \tag{B.29}$$

In the next sections we will include the dependence on F_{ij} on the boundary state of a space filling brane.

B.3 General solution for open strings: some technical details

In this section we solve the equation of motion and the boundary conditions in eq. (2.14) for an open string. To this purpose it is convenient to rewrite eq. (2.14) as follows:

$$\left[\mathcal{E}^T_{(0)ij}\partial_+ X^j - \mathcal{E}_{(0)ij}\partial_- X^j\right]_{\sigma=0} = 0 \tag{B.30}$$

and

$$\left[\mathcal{E}^T_{(\pi)ij}\partial_+ X^j - \mathcal{E}_{(\pi)ij}\partial_- X^j\right]_{\sigma=\pi} = 0 \tag{B.31}$$

where we have defined

$$\mathcal{E}_{(0)ij} \equiv G_{ij} - (B_{ij} - 2\pi\alpha' q_0 F^{(0)}_{ij}) = G_{ij} - \mathcal{B}_{(0)ij} = (E^T)_{ij} + \hat{F}_{ij} \tag{B.32}$$

with

$$\begin{aligned}\mathcal{B}_{(0)ij} &\equiv B_{ij} - 2\pi\alpha' q_0 F^{(0)}_{ij} = B_{ij} - q_0 \hat{F}^{(0)}_{ij} \\ \hat{F}^{(0)}_{ij} &\equiv 2\pi\alpha' F^{(0)}_{ij}\end{aligned} \tag{B.33}$$

and similarly for the (π) quantities.

The general solution of the bulk equation in (2.13) is given by:

$$X^i(\sigma,\tau) = \frac{1}{2}\left[G^{ij}\mathcal{E}_{(0)jk}F^k(\tau+\sigma) + G^{ij}(\mathcal{E}^T)_{(0)jk}G^k(\tau-\sigma)\right] \tag{B.34}$$

with $F^i(\tau+\sigma)$ and $G^i(\tau-\sigma)$ arbitrary functions.

We have chosen the particular form in eq. (B.34) because it immediately solves the boundary condition at $\sigma = 0$ as we will show shortly. By inserting eq. (B.34) in the two boundary conditions one gets:

$$(\mathcal{E}^T_{(0)} G^{-1}\mathcal{E}_{(0)})_{ij}\partial_\tau F^j(\tau) = (\mathcal{E}_{(0)}G^{-1}\mathcal{E}^T_{(0)})_{ij}\partial_\tau G^j(\tau) \tag{B.35}$$

and

$$(\mathcal{E}^T_{(\pi)} G^{-1}\mathcal{E}_{(0)})_{ij}\partial_\tau F^j(\tau+\pi) = (\mathcal{E}_{(\pi)}G^{-1}\mathcal{E}^T_{(0)})_{ij}\partial_\tau G^j(\tau-\pi). \tag{B.36}$$

Let us remind here that G^{ij} means the inverse of the matrix G_{ij}, i.e. $G^{ik}G_{kj} = \delta^i_j$. In the following we will denote G^{ij} with G^{-1} only when the indices i and j are not explicitly written. We are also using this convention for all other matrices. The boundary condition at $\sigma = 0$ is immediately solved by

$$G^i(\tau) = F^i(\tau) + const \tag{B.37}$$

since the open string metric $\mathcal{G}_{(0)}$ satisfies the relation:

$$\mathcal{G}_{(0)} = \mathcal{E}^T_{(0)} G^{-1} \mathcal{E}_{(0)} = \mathcal{E}_{(0)} G^{-1} \mathcal{E}^T_{(0)}.$$

In order to solve the boundary condition at $\sigma = \pi$ it is convenient to introduce the quantity:

$$R^i{}_j = \left((\mathcal{E}^T_{(\pi)} G^{-1} \mathcal{E}_{(0)})^{-1} \right)^{ik} \left(\mathcal{E}_{(\pi)} G^{-1} \mathcal{E}^T_{(0)} \right)_{kj} = \left(\mathcal{E}^{-1}_{(0)} G \mathcal{E}^{-T}_{(\pi)} \mathcal{E}_{(\pi)} G^{-1} \mathcal{E}^T_{(0)} \right)^i{}_j \tag{B.38}$$

which is a SO($\hat{d}$) matrix with respect to the metric $\mathcal{G}_{(0)}$

$$R^T \mathcal{G}_{(0)} R = \mathcal{G}_{(0)}.$$

The boundary condition at $\sigma = \pi$ can now be written as follows:

$$\partial_\tau F^i(\tau + \pi) = R^i{}_j \partial_\tau F^j(\tau - \pi). \tag{B.39}$$

In order to solve the previous equation one should diagonalize the R-matrix. However, in the dipole case, one can avoid such a problem because:

$$q_0 F^{(0)} - q_\pi F^{(\pi)} = 0 \Rightarrow R = \mathbb{I}.$$

In this case all the (0) quantities drop and one can simply write $\mathcal{E}$ for $\mathcal{E}_{(0)}$ and so on. The solution of eq. (B.39) is:

$$\partial_\tau F^i(\tau + \sigma) = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \left(e^{-i(\tau+\sigma)n\mathbb{I}} \right)^i{}_j \alpha^j_n \tag{B.40}$$

that can be integrated to give:

$$F^i(\tau + \sigma) = x^i + i\sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \left(\frac{1}{n} e^{-i(\tau+\sigma)n\mathbb{I}} \right)^i{}_j \alpha^j_n \tag{B.41}$$

where x^i is an arbitrary constant of integration. The open string expansion (excluding pure Dirichlet boundary conditions)[12]:

$$X^i(\sigma, \tau) = \frac{1}{2} \left(\hat{X}^i_L(\tau + \sigma) + \hat{X}^i_R(\tau - \sigma) \right) \tag{B.42}$$

where $\hat{X}^i_L(\tau + \sigma)$ and $\hat{X}^i_R(\tau - \sigma)$ are the ones already written respectively in (2.16) and (2.17). It is also useful to define the commuting coordinates x^i_0

$$x^i = x^i_0 - \pi\alpha' \Theta^{ij} \mathcal{G}_{jk} p^k \tag{B.43}$$

[12] With respect to the conventions used in [27] (CRS) and in [28] (C) we have $G = g_{CSR} = g_C, \mathcal{B} = -F_{CRS} = 2\pi\alpha' \mathcal{B}_C, \mathcal{G} = M_{CRS} = G_C, 2\pi\alpha'\Theta = \Theta_{CSR} = \Theta_C$.

where x_0 satisfies the usual commutation relations

$$[x_0^i, x_0^j] = 0 \quad ; \quad [x_0^i, p^j] = i\mathcal{G}^{ij} \tag{B.44}$$

Given the operator x_0, we define:

$$\begin{aligned}
\hat{X}^i_L(\tau+\sigma) &= \hat{X}^i_{L(0)}(\tau+\sigma) + \pi\alpha'(G^{-1}\mathcal{E}\Theta\mathcal{G})^i_j p^j \\
&= (G^{-1}\mathcal{E})^i_j \left(X^j_{L(0)}(\tau+\sigma) + \pi\alpha'(\Theta\mathcal{G})^j_l p^l\right) \\
&= (G^{-1}\mathcal{E})^i_j \left(X_{L(0)}(\tau+\sigma) + \pi\alpha' G^{-1}\mathcal{B}p\right)^j
\end{aligned} \tag{B.45}$$

and

$$\begin{aligned}
\hat{X}^i_R(\tau-\sigma) &= \hat{X}^i_{R(0)}(\tau-\sigma) + \pi\alpha'(G^{-1}\mathcal{E}^T\Theta\mathcal{G})^i_j p^j \\
&= (G^{-1}\mathcal{E}^T)^i_j \left(X^j_{R(0)}(\tau-\sigma) + \pi\alpha'(\Theta\mathcal{G})^i_j p^j\right) \\
&= (G^{-1}\mathcal{E}^T)^i_j \left(X_{R(0)}(\tau-\sigma) + \pi\alpha' G^{-1}\mathcal{B}p\right)^j
\end{aligned} \tag{B.46}$$

where all the quantities with (0) depend on x_0 instead of x. Here we have introduced

$$\begin{aligned}
X_{L(0)}(z) &= x_0 - 2\alpha' ip \ln z + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{sgn(n)}{\sqrt{|n|}} a_n z^{-n} \qquad 0 \leq arg(z) \leq \pi \\
X_{R(0)}(\bar{z}) &= x_0 - 2\alpha' ip \ln \bar{z} + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{sgn(n)}{\sqrt{|n|}} a_n \bar{z}^{-n} \quad -\pi \leq arg(\bar{z}) \leq 0
\end{aligned}$$

where $z = e^{i(\tau+\sigma)}$, Notice that for $\sigma = 0$ equations (B.42), (B.45) and (B.46) becomes

$$\begin{aligned}
X^i(\tau) &= \frac{1}{2}\left(\hat{X}^i_L(\tau) + \hat{X}^i_R(\tau)\right) \\
&= X_{L(0)}(\tau) - \pi\alpha'(\Theta\mathcal{G})^i_j p^j
\end{aligned} \tag{B.47}$$

where we have used that $X_{L(0)}(x) = X_{R(0)}(x)$.

The spectrum of p^i is given by

$$\mathcal{G}_{ij}p^j|k\rangle = k_i|k\rangle = \frac{n_i}{\sqrt{\alpha'}} - q_\pi a_{(\pi)i} + q_0 a_{(0)i}|k\rangle$$

with $n_i \in Z$ and where $a_i^{(0,\pi)}$ are the constant parts of the gauge fields $A_i^{(0,\pi)}$.

The OPEs read

$$\begin{aligned}
X_{L(0)}(z)X^T_{L(0)}(w) &= -2\alpha' \ln(z-w)\mathcal{G}^{-1} \\
X_{L(0)}(z)X^T_{R(0)}(\bar{w}) &= -2\alpha' \ln(z-\bar{w})\mathcal{G}^{-1} \\
X_{R(0)}(\bar{z})X^T_{R(0)}(\bar{w}) &= -2\alpha' \ln(\bar{z}-\bar{w})\mathcal{G}^{-1}
\end{aligned} \tag{B.48}$$

or using $\mathcal{E}\mathcal{G}^{-1}\mathcal{E}^T = \mathcal{E}^T\mathcal{G}^{-1}\mathcal{E} = G$

$$\begin{aligned}
\hat{X}_L(z)\hat{X}^T_L(w) &= -2\alpha' \ln(z-w)G^{-1} \\
\hat{X}_L(z)\hat{X}^T_R(\bar{w}) &= -2\alpha' \ln(z-\bar{w})G^{-1}\mathcal{E}\mathcal{G}^{-1}\mathcal{E}G^{-1} = -2\alpha' \ln(z-\bar{w})\mathcal{E}^{-T}\mathcal{E}G^{-1} \\
\hat{X}_R(\bar{z})\hat{X}^T_R(\bar{w}) &= -2\alpha' \ln(\bar{z}-\bar{w})G^{-1}
\end{aligned} \tag{B.49}$$

C. Short review of closed string canonical linear transformations

A general T-duality transformation is a canonical transformation of the form

$$\begin{pmatrix} \frac{X'^t}{2\pi\alpha'} \\ P^t \end{pmatrix} = \Lambda \begin{pmatrix} \frac{X'}{2\pi\alpha'} \\ P \end{pmatrix} \tag{C.1}$$

with

$$\Lambda = \begin{pmatrix} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{pmatrix} \in O(\hat{d}, \hat{d}, Z) \tag{C.2}$$

where $X = \parallel X^i \parallel$ and $P = \parallel P_i \parallel$ with $i = 1, \dots \hat{d}$ are column vectors. Here and in what follows, the momentum p is understood with covariant indices, unless explicitly indicated. To belong to the group $O(\hat{d}, \hat{d}, Z)$ the matrix Λ must be a $\hat{d} \times \hat{d}$ matrix with integer entries satisfying the constraint

$$\Lambda \begin{pmatrix} 0 & 1_{\hat{d}} \\ 1_{\hat{d}} & 0 \end{pmatrix} \Lambda^T = \begin{pmatrix} 0 & 1_{\hat{d}} \\ 1_{\hat{d}} & 0 \end{pmatrix} \equiv J \tag{C.3}$$

This constraint simply follows from the canonical commutation relations:

$$[X'(\sigma), P^T(\sigma')] = [P(\sigma), {X'}^T(\sigma')] = i\partial_\sigma \delta(\sigma - \sigma')1_{\hat{d}}$$

which imply

$$\left[\begin{pmatrix} \frac{X'}{2\pi\alpha'} \\ P \end{pmatrix} (\sigma), \left(\frac{X'^T}{2\pi\alpha'}, P^T \right)(\sigma') \right] = \begin{pmatrix} 0 & 1_{\hat{d}} \\ 1_{\hat{d}} & 0 \end{pmatrix} \frac{i}{2\pi\alpha'} \partial_\sigma \delta(\sigma - \sigma'). \tag{C.4}$$

Under the transformation in eq. (C.1) the previous commutator becomes

$$\left[\begin{pmatrix} \frac{X'^t}{2\pi\alpha'} \\ P^t \end{pmatrix} (\sigma), \left(\frac{(X'^t)^T}{2\pi\alpha'}, P^{tT} \right)(\sigma') \right] = \Lambda \begin{pmatrix} 0 & 1_{\hat{d}} \\ 1_{\hat{d}} & 0 \end{pmatrix} \Lambda^T \frac{i}{2\pi\alpha'} \partial_\sigma \delta(\sigma - \sigma') \tag{C.5}$$

However the commutation relation in (C.4) has to be invariant under the transformation in eq. (C.1) and thus equating the left hand sides of eqs. (C.4) and (C.5) one gets eq. (C.3) implying $\Lambda \in O(\hat{d}, \hat{d}, R)$. In order to derive the constraint $\Lambda \in O(\hat{d}, \hat{d}, Z)$ for the T-duality group we have to work a little more and we first define:

$$\partial \equiv \frac{\partial}{\partial(\sigma + \tau)} \qquad \bar{\partial} \equiv \frac{\partial}{\partial(\tau - \sigma)}$$

and then we get:

$$X' = \partial X_L(\tau + \sigma) - \bar{\partial}\tilde{X}_R(\tau - \sigma) \qquad 2\pi\alpha' P = E\partial X_L + E^T \bar{\partial}\tilde{X}_R$$

with $E_{ij} = G_{ij} + B_{ij}$, being X_L and $\tilde{X}_R$ defined in eqs. (B.10) and (B.11). eqs. (C.1) can be split into an holomorphic and an antiholomorphic part as

$$\begin{pmatrix} \partial X_L^t \\ E^t \partial X_L^t \end{pmatrix} = \Lambda \begin{pmatrix} \partial X_L \\ E\partial X_L \end{pmatrix} \qquad \begin{pmatrix} -\bar{\partial}\tilde{X}_R^t \\ E^{tT}\bar{\partial}\tilde{X}_R^t \end{pmatrix} = \Lambda \begin{pmatrix} -\bar{\partial}\tilde{X}_R \\ E^T \bar{\partial}\tilde{X}_R \end{pmatrix} \tag{C.6}$$

Looking first at the zero modes we get the following equations for the left and right momenta

$$\left\{\begin{array}{rl}(p^t_L)^i = & \left(\hat{\mathcal{A}}+\hat{\mathcal{B}}E\right)^i{}_j\, p^j_L \\ (E^t p^t_L)_i = & \left(\hat{\mathcal{C}}+\hat{\mathcal{D}}E\right)_{ij}\, p^j_L \end{array}\right. \qquad \left\{\begin{array}{rl}(p^t_R)^i = & \left(\hat{\mathcal{A}}-\hat{\mathcal{B}}E^T\right)^i{}_j\, p^j_R \\ (E^{tT} p^t_R)_i = & \left(-\hat{\mathcal{C}}+\hat{\mathcal{D}}E^T\right)_{ij}\, p^j_R \end{array}\right. \tag{C.7}$$

and, if we remember that windings and momenta in compact space are defined as

$$m = \sqrt{\alpha'}G^{-1}(p_L - p_R) \qquad n = \sqrt{\alpha'}(EG^{-1}p_L + E^T G^{-1} p_R) \tag{C.8}$$

where $m = \parallel m^i \parallel$ and $n = \parallel n_i \parallel$ and the momenta are understood with covariant indeces, we easily get that under the transformation in eq. (C.7) m and n transform as

$$\left\{\begin{array}{rl} m^t = & \hat{\mathcal{A}}m + \hat{\mathcal{B}}n \\ n^t = & \hat{\mathcal{C}}m + \hat{\mathcal{D}}n \end{array}\right. \leftrightarrow \begin{pmatrix} m^t \\ n^t \end{pmatrix} = \Lambda \begin{pmatrix} m \\ n \end{pmatrix} \tag{C.9}$$

which implies the desired constraint, i.e. $\Lambda \in O(\hat{d},\, \hat{d},\, Z)$.

If we now consider the other terms we get the following equations for the left and right oscillators

$$\left\{\begin{array}{rl} \alpha^t_n = & \left(\hat{\mathcal{A}}+\hat{\mathcal{B}}E\right)\alpha_n \\ \tilde{\alpha}^t_n = & \left(\hat{\mathcal{A}}-\hat{\mathcal{B}}E^T\right)\tilde{\alpha}_n \end{array}\right. \qquad n \in Z^* \tag{C.10}$$

For the sake of completeness we collect here some consequences of eq. (C.3). We find the expression for the inverse transformation matrices Λ^{-1} and Λ^{-T} to be

$$\Lambda^{-1} = \begin{pmatrix} 0 & 1_{\hat{d}} \\ 1_{\hat{d}} & 0 \end{pmatrix} \Lambda^T \begin{pmatrix} 0 & 1_{\hat{d}} \\ 1_{\hat{d}} & 0 \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{D}}^T & \hat{\mathcal{B}}^T \\ \hat{\mathcal{C}}^T & \hat{\mathcal{A}}^T \end{pmatrix} \tag{C.11}$$

$$\Lambda^{-T} = \begin{pmatrix} 0 & 1_{\hat{d}} \\ 1_{\hat{d}} & 0 \end{pmatrix} \Lambda \begin{pmatrix} 0 & 1_{\hat{d}} \\ 1_{\hat{d}} & 0 \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{D}} & \hat{\mathcal{C}} \\ \hat{\mathcal{B}} & \hat{\mathcal{A}} \end{pmatrix} \tag{C.12}$$

so that eq. (C.3) can be explicitly written as

$$\Lambda\, J\, \Lambda^T = \begin{pmatrix} \hat{\mathcal{B}}\hat{\mathcal{A}}^T + \hat{\mathcal{A}}\hat{\mathcal{B}}^T & \hat{\mathcal{B}}\hat{\mathcal{C}}^T + \hat{\mathcal{A}}\hat{\mathcal{D}}^T \\ \hat{\mathcal{D}}\hat{\mathcal{A}}^T + \hat{\mathcal{C}}\hat{\mathcal{B}}^T & \hat{\mathcal{D}}\hat{\mathcal{C}}^T + \hat{\mathcal{C}}\hat{\mathcal{D}}^T \end{pmatrix} = \begin{pmatrix} 0 & 1_{\hat{d}} \\ 1_{\hat{d}} & 0 \end{pmatrix} \tag{C.13}$$

$$\Lambda^T\, J\, \Lambda = \begin{pmatrix} \hat{\mathcal{A}}^T\hat{\mathcal{C}} + \hat{\mathcal{C}}^T\hat{\mathcal{A}} & \hat{\mathcal{A}}^T\hat{\mathcal{D}} + \hat{\mathcal{C}}^T\hat{\mathcal{B}} \\ \hat{\mathcal{B}}^T\hat{\mathcal{C}} + \hat{\mathcal{D}}^T\hat{\mathcal{A}} & \hat{\mathcal{B}}^T\hat{\mathcal{D}} + \hat{\mathcal{D}}^T\hat{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} 0 & 1_{\hat{d}} \\ 1_{\hat{d}} & 0 \end{pmatrix} \tag{C.14}$$

where in the second equality we have used the following identity:

$$\Lambda\, J\, \Lambda^T = J \Rightarrow \Lambda\, J\, \Lambda^T\, J = \mathbb{I} \Rightarrow \Lambda\, J\, \Lambda^T\, J\, \Lambda = \Lambda \Rightarrow \Lambda^T\, J\, \Lambda = J$$

The inverses of eqs. (C.9) and (C.10) can then be obtained using eq. (C.11), i.e $\hat{\mathcal{A}} \leftrightarrow \hat{\mathcal{D}}^T, \hat{\mathcal{B}} \to \hat{\mathcal{B}}^T, \hat{\mathcal{C}} \to \hat{\mathcal{C}}^T$ and exchanging ^{t} quantities with those without ^{t}, explicitly

$$\left\{\begin{array}{rl} m = & \hat{\mathcal{D}}^T m^t + \hat{\mathcal{B}}^T n^t \\ n = & \hat{\mathcal{C}}^T m^t + \hat{\mathcal{A}}^T n^t \end{array}\right.$$

$$\begin{cases} \alpha_n = \left(\hat{\mathcal{D}}^T + \hat{\mathcal{B}}^T E^t\right) \alpha_n^t \\ \tilde{\alpha}_n = \left(\hat{\mathcal{D}}^T - \hat{\mathcal{B}}^T E^{tT}\right) \tilde{\alpha}_n^t \end{cases} \quad n \in Z^* \tag{C.15}$$

From the two eqs. (C.6) we get two different expressions for the relation between E^t and E

$$E^{tT} = (-\hat{\mathcal{C}} + \hat{\mathcal{D}}E^T)(\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)^{-1} \quad E^t = (\hat{\mathcal{C}} + \hat{\mathcal{D}}E)(\hat{\mathcal{A}} + \hat{\mathcal{B}}E)^{-1} \tag{C.16}$$

which are compatible because of eqs. (C.14).

Finally we give the transformation properties of the background metric

$$G^t = (\hat{\mathcal{A}} + \hat{\mathcal{B}}E)^{-T} G (\hat{\mathcal{A}} + \hat{\mathcal{B}}E)^{-1} = (\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)^{-T} G (\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)^{-1} \tag{C.17}$$

which can be easily shown by writing $G^t = E^t + E^{tT}$ and using respectively the second eq. in (C.16) with its transposed and the first Equation in (C.16) with its transposed.

Finally it is useful to establish the connection between our notation and the one used in [38]. To this purpose we must consider the closed string Hamiltonian given in eq. (B.16). By requiring the Hamiltonian to be invariant under the $O(\hat{d}, \hat{d}, Z)$ transformation

$$\frac{H^t}{2} = \frac{1}{2}(Z^t)^T M^t Z^t + \cdots = \frac{1}{2}(\Lambda Z)^T \Lambda^{-T} M \Lambda^{-1} (\Lambda Z) + \cdots \equiv \frac{H}{2}$$

one gets $M^t = \Lambda^{-T} M \Lambda^{-1}$. By comparing such transformation with eq. (2.4.19) of ref. [38] we get that $g = \Lambda^{-T}$. By using the expressions of g as given in ref. [38] and Λ as defined in eq. (C.3), we get:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{D}} & \hat{\mathcal{C}} \\ \hat{\mathcal{B}} & \hat{\mathcal{A}} \end{pmatrix}$$

where we have used the identity $\Lambda^{-T} = J\Lambda J$, which trivially follows from eq. (C.3). In the last part of this appendix we give some relations useful to determine the normalization of the boundary state given in eq. (4.4). To this aim we notice that:

$$\begin{aligned} \frac{(\det E^t)^2}{\det G^t} &= \left(\frac{\det(-\hat{\mathcal{C}} + \hat{\mathcal{D}}E^T)}{\det(\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)}\right)^2 \frac{\det(\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)^T \det(\hat{\mathcal{A}} - \hat{\mathcal{B}}E^T)}{\det G} \\ &= \begin{cases} (\det(-\hat{\mathcal{C}} + \hat{\mathcal{D}}E^T))^2 \frac{1}{\det G} & \det \hat{\mathcal{D}} = 0 \\ (\det \hat{\mathcal{D}})^2 (\det \mathcal{E})^2 \frac{1}{\det G} & \det \hat{\mathcal{D}} \neq 0 \end{cases} \end{aligned} \tag{C.18}$$

where we have used the first equation in (C.16) and the second in (C.17) together with the equality

$$\det \hat{\mathcal{D}}^{-1} \det(-\hat{\mathcal{C}} + \hat{\mathcal{D}}E^T) = \det(F + E^T) = \det \mathcal{E}$$

which follows from eqs. (4.3) (B.32). In this way we have determined the zero mode part of the boundary state. Moving to the non-zero modes we notice that

$$\begin{aligned} \alpha_{-n}^{tT} E^{tT} E^{t-1} G^t \tilde{\alpha}_{-n}^t &= \alpha_{-n}^T (\hat{\mathcal{C}} + \hat{\mathcal{D}}E)^T (-\hat{\mathcal{C}} + \hat{\mathcal{D}}E^T)^{-T} G \alpha_{-n}^t \\ &\Rightarrow \alpha_{-n}^T \mathcal{E}\mathcal{E}^{-T} G \tilde{\alpha}_{-n} \quad \det \hat{\mathcal{D}} \neq 0 \end{aligned} \tag{C.19}$$

where we have used eqs. (C.10) and (C.16) together with the following identity:

$$(\hat{\mathcal{C}} + \hat{\mathcal{D}}E)^T (-\hat{\mathcal{C}} + \hat{\mathcal{D}}E^T)^{-T} = (\hat{\mathcal{D}}^{-1}\hat{\mathcal{C}} + E)^T (-\hat{\mathcal{D}}^{-1}\hat{\mathcal{C}} + E^T)^{-T} = \mathcal{E}\mathcal{E}^{-T}.$$

D. Boundary state: closed string calculation

In this appendix we determine directly in the closed string channel, the compact part of the boundary state describing a non-abelian brane compactified on T^2 and in the presence of a background gauge field with constant field strength. The generalization of such calculation to a generic torus T^6 will be trivial. In order to simplify the calculation, we take the background gauge field in the gauge:

$$A_1 = 0 \qquad A_2 = F_{12}x^1$$

which is different from the gauge choice made in the section 2. Here, the x^i are the compact coordinates bounded between 0, $2\pi\sqrt{\alpha'}$, i.e. $0 \leq x^i < 2\pi\sqrt{\alpha'}$, with $i = 1, 2$.

The boundary state in the presence of a magnetic field is related to the uncharged one by the relation [33]:

$$|D25(E, F)\rangle = Tr\left(P\ e^{+i\oint qA}\right)\ |D25(E, F = 0)\rangle \tag{D.1}$$

Denoting by γ the closed path of the integration, we parameterized it as follows:

$$\gamma : \quad \sigma \in [0,\, \pi] \longrightarrow (X^1(\sigma),\, X^2(\sigma))$$

The path, in general, will wrap w^i times the torus and the details of such a wrapping are important in the evaluation of the path-ordering appearing in the eq. (D.1). This is because every time that the curve makes a turn around the cycles of the torus, the gauge transition functions must be introduced "to glue" the fields at the boundaries of the torus. Such a gluing can be realized as follows. We first choose the origin of the compact frame coincident with the first end of the curve and label with λ_i, ($\lambda_0 = 0$ and $\lambda_{M+1} = \pi$) the values which the parameter σ takes when the path cross the boundary values $x^i = 0,\, 2\pi\sqrt{\alpha'}$, $i = 1, 2$. We can write then:

$$X^i(\sigma) = x^i(\sigma) + 2\pi\sqrt{\alpha'}\sum_{k=1}^{p} s^i_k \quad ; \quad \sigma \in [\lambda_p,\, \lambda_{p+1}]$$

Here $0 \leq x^i(\sigma) < 2\pi\sqrt{\alpha'}$ and $s^i_p = -1, 1$ respectively if the path in the corresponding interval $[\lambda_{p-1}, \lambda_p]$ "unwraps" or wraps once, while is zero if the curve is constant in the interval. The total wrapping will be given by:

$$w^i = \sum_{k=1}^{M} s^i_k$$

Now, we are ready to explicitly compute the path-ordering introduced in eq. (D.1) in the case of non-abelian branes. It is given by:

$$\mathrm{Tr}\left(P\ e^{i\oint qA}\right) = \mathrm{Tr}\left[e^{i\int_{\lambda_0}^{\lambda_1} q\,F_{12}x^1x'^2 d\tau}\,\Omega_2^{s^2_1}\Omega_1^{s^1_1}\ldots e^{i\int_{\lambda_p}^{\lambda_{p+1}} q\,F_{12}x^1x'^2 d\tau}\,\Omega_2^{s^2_{p+1}}\Omega_1^{s^1_{p+1}}\ldots\right]$$

being, in this gauge, the $\mathrm{U}(1)$ factor of the gauge transition function slightly different from the one given in section 2:

$$\Omega_1 = e^{-2\pi i\sqrt{\alpha'}q\, F_{12}x^2}\omega_1 \qquad \Omega_2 = \omega_2$$

By using the previous parametrization of the curve γ, we can write:

$$\begin{aligned} e^{i\int_{\lambda_p}^{\lambda_{p+1}} q\, F_{12}x^1 x'^2 d\sigma}\Omega_2^{s^2_{p+1}}\Omega_1^{s^1_{p+1}} &= e^{i\int_{\lambda_p}^{\lambda_{p+1}} q\, F_{12}X^1X'^2 d\sigma - 2\pi i\sqrt{\alpha'}\sum_{k=1}^{p} q\, F_{12}s^1_k(X^2(\lambda_{p+1})-X^2(\lambda_p))} \\ &\quad\times e^{-2\pi i\sqrt{\alpha'}s^1_{p+1}q\, F_{12}(X^2(\lambda_{p+1})-2\pi\sqrt{\alpha'}\sum_{k=1}^{p+1}s^2_k)}\omega_2^{s^2_{p+1}}\omega_1^{s^1_{p+1}} \end{aligned}$$

which implies:

$$\begin{aligned} P\ e^{i\oint qA} &= e^{i\int_{\lambda_0}^{\lambda_{M+1}} q\, F_{12}X^1X'^2 d\sigma - 2\pi i\sqrt{\alpha'}\sum_{p=1}^{M}\sum_{k=1}^{p} q\, F_{12}s^1_k(X^2(\lambda_{p+1})-X^2(\lambda_p))} \\ &\quad\times e^{-2\pi i\sqrt{\alpha'}\sum_{p=0}^{M-1} s^1_{p+1}q\, F_{12}(X^2(\lambda_{p+1})-2\pi\sqrt{\alpha'}\sum_{k=1}^{p+1}s^2_k)} \\ &\quad\times\omega_2^{s^2_1}\,\omega_1^{s^1_1}\,\omega_2^{s^2_2}\,\omega_1^{s^1_2}\ldots\omega_2^{s^2_M}\,\omega_1^{s^1_M} \end{aligned} \tag{D.2}$$

The reordering of the last factor gives:

$$\omega_2^{s^2_1}\,\omega_1^{s^1_1}\,\omega_2^{s^2_2}\,\omega_1^{s^1_2}\ldots\omega_2^{s^2_M}\,\omega_1^{s^1_M} = e^{-(2\pi\sqrt{\alpha'})^2 iF_{12}\sum_{p=1}^{M}s^1_p\sum_{k=1}^{p}s^2_k}\omega_1^{w^1}\ \omega_2^{w^2}$$

which cancels the last factor in the second line of the eq. (D.2), while observing that:

$$\sum_{p=1}^{M}\left[\sum_{k=1}^{p}s^1_k(X^2(\lambda_{p+1})-X^2(\lambda_p))+s^1_pX^2(\lambda_p)\right] = w^1\,X^2(\pi)$$

we can write:

$$\mathrm{Tr}\left(P\ e^{i\oint qA}\right) = e^{i\int_0^{\pi} q\, F_{12}X^1X'^2 d\sigma}e^{-2\pi i\sqrt{\alpha'}F_{12}w^1\,X^2(\pi)}\mathrm{Tr}\left[\omega_1^{w^1}\ \omega_2^{w^2}\right]$$

In particular the path ordering must be evaluated on the string coordinate expansion with the result

$$P\ e^{i\oint qA} = e^{i2\pi q\left[F_{ij}x^i\hat{m}^j-\pi F_{ij}\hat{m}^i\hat{m}^j\right]}e^{-\pi\alpha' qF_{ij}\sum_{n=1}^{\infty}(a^i_n-\tilde{a}^i_{-n})(a^j_{-n}-\tilde{a}^j_n)}\omega_1^{\hat{m}^1}\ \omega_2^{\hat{m}^2}$$

where now all $\hat{m}^i$ are winding operators.

Let us now determine the non-zero modes contribution of the boundary state, starting from the expression given in eq. (B.28). The latter corresponds to evaluate the following product of operators:

$$e^{-\pi\alpha' qF_{ij}(a^i-\tilde{a}^{i\dagger})(a^{j\dagger}-\tilde{a}^j)}e^{-\ a^{i\dagger}G_{ih}(E^{-1})^{hk}(E^T)_{kj}\tilde{a}^{j\dagger}}|0>$$

which, since $(a^i-\tilde{a}^{i\dagger})$ and $(a^{j\dagger}-\tilde{a}^j)$ commute, can be easily evaluated with the introduction of an auxiliary variable z:

$$e^{-\pi\alpha' qF_{ij}(a^i-\tilde{a}^{i\dagger})(a^{j\dagger}-\tilde{a}^j)} = \int\prod_i\frac{dz_i\ d\bar{z}^i}{\pi}e^{-z_i\bar{z}^i+\pi\alpha' qF_{ij}(a^i-\tilde{a}^{i\dagger})\bar{z}^j-(a^{i\dagger}-\tilde{a}^i)z_i}$$

JHEP11(2007)100

The previous integral can be performed and one gets:

$$\int \prod_i \frac{dz_i\ d\bar{z}^i}{\pi} e^{-z_i\bar{z}^i + \pi\alpha' q F_{ij}(a^i - \tilde{a}^{i\dagger})\bar{z}^j - (a^{i\dagger} - \tilde{a}^i)z_i}\ e^{-\ a^{i\dagger}G_{ih}(E^{-1})^{hk}(E^T)_{kj}\tilde{a}^{j\dagger}}|0>$$
$$= \int \prod_i \frac{dz_i\ d\bar{z}^i}{\pi} e^{-z_i\bar{z}^i} e^{\pi\alpha' q F_{ij} a^i \bar{z}^j} e^{-\pi\alpha' q F_{ij}\tilde{a}^{i\dagger}\bar{z}^j} e^{-a^{i\dagger}z_i} e^{\tilde{a}^i z_i}\ e^{-\ a^{i\dagger}G_{ih}(E^{-1})^{hk}(E^T)_{kj}\tilde{a}^{j\dagger}}|0>$$
$$= \int \prod_i \frac{dz_i\ d\bar{z}^i}{\pi} e^{-z_i\bar{z}^i}$$
$$e^{\pi\alpha' q (G^{-1})^{in}F_{nl}\bar{z}^l\left[-G_{ih}(E^{-1})^{hk}(E^T)_{kj}\tilde{a}^{j\dagger} - \left(G_{ih}(E^{-1})^{hk}(E^T)_{kj}(G^{-1})^{mj} + \delta_i^j\right)z_j\right]}$$
$$e^{-a^{i\dagger}G_{ih}(E^{-1})^{hk}(E^T)_{kj}\tilde{a}^{j\dagger} - a^{i\dagger}\left(G_{ih}(E^{-1})^{hk}(E^T)_{km}(G^{-1})^{mj} + \delta_i^j\right)z_j - \pi\alpha' q F_{ij}\bar{z}^j\tilde{a}^{i\dagger}}|0>$$
$$= \int \prod_i \frac{dz_i\ d\bar{z}^i}{\pi} e^{-z_i\bar{z}^i} e^{\pi\alpha' q (G^{-1})^{in}F_{nl}\bar{z}^l\left[-G_{ih}(E^{-1})^{hk}(E^T)_{kj}\tilde{a}^{j\dagger} - 2G_{ih}(E^{-1})^{hj}z_j\right]}$$
$$e^{-a^{i\dagger}G_{ih}(E^{-1})^{hk}(E^T)_{kj}\tilde{a}^{j\dagger} - 2a^{i\dagger}G_{ih}(E^{-1})^{hj}z_j - \pi\alpha' q F_{ij}\bar{z}^j\tilde{a}^{i\dagger}}|0>$$
$$= \int \prod_i \frac{dz_i\ d\bar{z}^i}{\pi} e^{-z_i\left[\delta_j^i - 2\pi\alpha' q F_{jh}(E^{-1})^{hi}\right]\bar{z}^j}\ e^{2\pi\alpha' q \bar{z}^j F_{jh}(E^{-1})^{hk}G_{ki}\tilde{a}^{i\dagger}} e^{-2a^{i\dagger}G_{ih}(E^{-1})^{hj}z_j}$$
$$e^{-a^{i\dagger}G_{ih}(E^{-1})^{hk}(E^T)_{kj}\tilde{a}^{j\dagger}}|0>$$
$$= \left[\det(\delta_j^i - 2\pi\alpha' q(FE^{-1})_i^j\right]^{-1} e^{-2\ a^{i\dagger}G_{ih}(\mathcal{E}^{-T})^{hk}(\hat{F})_{kl}(E^{-1})^{lm}G_{mj}\tilde{a}^{j\dagger}}$$
$$e^{-\ a^{i\dagger}G_{ih}(E^{-1})^{hk}(E^T)_{kj}\tilde{a}^{j\dagger}}|0>$$
$$= \left[\det(\mathcal{E}^T E^{-1})_i^j\right]^{-1} e^{-\ a^{i\dagger}G_{ih}(\mathcal{E}^{-T})^{hk}(\mathcal{E})_{kj}\tilde{a}^{j\dagger}}|0>$$

Then, by using the zeta function regularization $\sum_1^\infty 1 = \zeta(0) = -\frac{1}{2}$, we can get the complete contribution from non zero-mode

$$\left[Tr\left(P\ e^{i\oint qA}\right)|D25(E,F=0)\rangle\right]_{nzm} = \sqrt{\det(\mathcal{E}^T E^{-1})} e^{-\sum_{n=1}^\infty\ a_n^{i\dagger}(G\mathcal{E}^{-T}\mathcal{E})_{ij}\tilde{a}^{j\dagger n}}|0>$$

We can now examine the zero modes contribution and we find

$$\left[Tr\left(P\ e^{i\oint qA}\right)\ |D25(E,F=0)\rangle\right]_{zm} =$$
$$= \frac{\sqrt{\det E}}{(\det G)^{1/4}} \sum_s Tr(\omega_1^{s_1}\omega_2^{s_2})\ e^{-i\pi\hat{F}_{12}s^1s^2}|n_i = \hat{F}_{ij}s^j, m^i = s^i\rangle$$

The previous calculation can also be generalized to a generic torus, getting:

$$\left[Tr\left(P\ e^{i\oint qA}\right)\ |D25(E,F=0)\rangle\right]_{zm} =$$
$$= \frac{\sqrt{\det E}}{(\det G)^{1/4}} \sum_s Tr(\omega_1^{s_1}\omega_2^{s_2}\ldots\omega_{\hat{d}}^{s_{\hat{d}}})\ e^{-i\pi\hat{F}_{ij}^< s^i s^j}|n_i = \hat{F}_{ij}s^j, m^i = s^i\rangle$$

where $\hat{F}_{ij}^< = \hat{F}_{ij}$ if $i<j$, zero otherwise. The factor $Tr(\omega_1^{s_1}\ldots\omega_{\hat{d}}^{s_{\hat{d}}})$ acts as a projector on the possible values of the integers s. This projector depends explicitly on the form of the various ω but we can nevertheless deduce the important constraints

$$n_i = \hat{F}_{ij}s^j \in Z$$

which are valid for all the values of s which survive the projection. The proof is very easy and for n_1 goes as

$$\begin{aligned} Tr(\omega_1^{s_1}\dots\omega_{\hat d}^{s_{\hat d}}) &= Tr(\omega_1\omega_1^{s_1}\dots\omega_{\hat d}^{s_{\hat d}}\omega_1^{-1}) \\ &= e^{i2\pi\hat F_{1j}s^j}Tr(\omega_1^{s_1}\dots\omega_{\hat d}^{s_{\hat d}}) \end{aligned}$$

The final form of the boundary reads

$$\begin{aligned} |D25(E,F)\rangle &= \frac{T_{25}}{2}\, N\frac{\sqrt{\det\mathcal{E}}}{(\det G)^{1/4}}\sum_s \frac{Tr(\omega_1^{s_1}\omega_2^{s_2}\dots\omega_{\hat d}^{s_{\hat d}})}{N} e^{-i\pi\hat F_{ij}^{<}s^is^j}|n_i=\hat F_{ij}s^j, m^i=s^i\rangle \\ &\quad\times e^{-\ a^{i\dagger}G_{ih}(\mathcal{E}^{-T})^{hk}(\mathcal{E})_{kj}\tilde a^{j\dagger}}|0> \end{aligned} \tag{D.3}$$

with:

$$Tr(\omega_1^{s_1}\omega_2^{s_2}\dots\omega_{\hat d}^{s_{\hat d}}) = N\,\delta_{s_1,0}^{[N]}\dots\delta_{s_{\hat d},0}^{[N]}$$

References

[1] A.M. Uranga, *Chiral four-dimensional string compactifications with intersecting D-branes*, *Class. and Quant. Grav.* **20** (2003) S373 [hep-th/0301032].

[2] D. Lüst, *Intersecting brane worlds: a path to the standard model?*, *Class. and Quant. Grav.* **21** (2004) S1399 [hep-th/0401156].

[3] R. Blumenhagen, M. Cvetič, P. Langacker and G. Shiu, *Toward realistic intersecting D-brane models*, *Ann. Rev. Nucl. Part. Sci.* **55** (2005) 71 [hep-th/0502005].

[4] R. Blumenhagen, B. Körs, D. Lüst and S. Stieberger, *Four-dimensional string compactifications with D-branes, orientifolds and fluxes*, *Phys. Rept.* **445** (2007) 1 [hep-th/0610327].

[5] W. Taylor, *Lectures on D-branes, gauge theory and M(atrices)*, hep-th/9801182.

[6] A. Hashimoto and W. Taylor, *Fluctuation spectra of tilted and intersecting D-branes from the Born-Infeld action*, *Nucl. Phys.* **B 503** (1997) 193 [hep-th/9703217].

[7] Z. Guralnik and S. Ramgoolam, *Torons and D-brane bound states*, *Nucl. Phys.* **B 499** (1997) 241 [hep-th/9702099].

[8] A. Hashimoto, *Perturbative dynamics of fractional strings on multiply wound d-strings*, *Int. J. Mod. Phys.* **A 13** (1998) 903 [hep-th/9610250].

[9] R. Rabadán, *Branes at angles, torons, stability and supersymmetry*, *Nucl. Phys.* **B 620** (2002) 152 [hep-th/0107036].

[10] D. Cremades, L.E. Ibáñez and F. Marchesano, *Computing Yukawa couplings from magnetized extra dimensions*, *JHEP* **05** (2004) 079 [hep-th/0404229].

[11] F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, *Noncommutative geometry from strings and branes*, *JHEP* **02** (1999) 016 [hep-th/9810072].

[12] R. Blumenhagen, L. Görlich, B. Körs and D. Lüst, *Asymmetric orbifolds, noncommutative geometry and type-I string vacua*, *Nucl. Phys.* **B 582** (2000) 44 [hep-th/0003024].

JHEP11(2007)100

[13] R. Blumenhagen, L. Görlich, B. Körs and D. Lüst, *Noncommutative compactifications of type-I strings on tori with magnetic background flux*, *JHEP* **10** (2000) 006 [hep-th/0007024].

[14] M. Bianchi and E. Trevigne, *The open story of the magnetic fluxes*, *JHEP* **08** (2005) 034 [hep-th/0502147].

[15] I. Antoniadis, A. Kumar and T. Maillard, *Magnetic fluxes and moduli stabilization*, *Nucl. Phys.* **B 767** (2007) 139 [hep-th/0610246].

[16] I. Antoniadis, A. Kumar and B. Panda, *Supersymmetric* SU(5) *GUT with stabilized moduli*, arXiv:0709.2799.

[17] I. Pesando, *Boundary states for branes with non trivial homology in constant closed and open background*, hep-th/0505052.

[18] P. Di Vecchia, A. Liccardo, R. Marotta, F. Pezzella and I. Pesando, *Boundary state for magnetized d9 branes and one-loop calculation*, hep-th/0601067.

[19] M. Bertolini, M. Billó, A. Lerda, J.F. Morales and R. Russo, *Brane world effective actions for D-branes with fluxes*, *Nucl. Phys.* **B 743** (2006) 1 [hep-th/0512067].

[20] D. Cremades, L.E. Ibáñez and F. Marchesano, *Yukawa couplings in intersecting D-brane models*, *JHEP* **07** (2003) 038 [hep-th/0302105].

[21] M. Cvetič and I. Papadimitriou, *Conformal Field Theory couplings for intersecting D-branes on orientifolds*, *Phys. Rev.* **D 68** (2003) 046001 [*Erratum ibid.* **70** (2004) 029903] [hep-th/0303083].

[22] S.A. Abel and A.W. Owen, *Interactions in intersecting brane models*, *Nucl. Phys.* **B 663** (2003) 197 [hep-th/0303124].

[23] R. Russo and S. Sciuto, *The twisted open string partition function and Yukawa couplings*, *JHEP* **04** (2007) 030 [hep-th/0701292].

[24] D. Duo, R. Russo and S. Sciuto, *New twist field couplings from the partition function for multiply wrapped D-branes*, hep-th/07091805.

[25] G. 't Hooft, *Some twisted selfdual solutions for the Yang-Mills equations on a hypertorus*, *Commun. Math. Phys.* **81** (1981) 267.

[26] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219]; *Constrained quantization of open string in background b field and noncommutative D-brane*, *Nucl. Phys.* **B 568** (2000) 447 [hep-th/9906192].

[27] C.-S. Chu, R. Russo and S. Sciuto, *Multiloop string amplitudes with B-field and noncommutative QFT*, *Nucl. Phys.* **B 585** (2000) 193 [hep-th/0004183].

[28] C.-S. Chu, *Non-commutative geometry from strings*, hep-th/0502167.

[29] L. Görlich, $N = 1$ *and non-supersymmetric open string theories in six and four space-time dimensions*, hep-th/0401040.

[30] E.J. Ferrer and V. de la Incera, *Mass eigenvalues of the open charged string in a magnetic background*, *Phys. Rev.* **D 52** (1995) 1011; *Global symmetries of open strings in an electromagnetic background*, *Phys. Rev.* **D 49** (1994) 2926.

[31] J. Polchinski, S. Chaudhuri and C.V. Johnson, *Notes on D-branes*, hep-th/9602052.

[32] M. Bianchi and E. Trevigne, *Gauge thresholds in the presence of oblique magnetic fluxes*, *JHEP* **01** (2006) 092 [hep-th/0506080].

[33] C.G. Callan, C. Lovelace, C.R. Nappi and S.Y. Yost, *Loop corrections to superstring equation of motion*, *Nucl. Phys.* **B 308** (1988) 221.

[34] C. Angelantonj, M. Cardella and N. Irges, *Scherk-Schwarz breaking and intersecting branes*, *Nucl. Phys.* **B 725** (2005) 115 [hep-th/0503179].

[35] M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, *Scattering of closed strings from many D-branes*, *Phys. Lett.* **B 400** (1997) 52 [hep-th/9702037].

[36] I. Pesando, *Multibranes boundary states with open string interactions*, to be published in Nucl. Phys. B, hep-th/0310027.

[37] M. Ademollo et al., *Soft dilations and scale renormalization in dual theories*, *Nucl. Phys.* **B 94** (1975) 221.

[38] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rept.* **244** (1994) 77 [hep-th/9401139].